

Existence of superconducting solutions for a reduced Ginzburg-Landau model in the presence of strong electric currents.

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Abstract

In this work we consider a reduced Ginzburg-Landau model in which the magnetic field is neglected and the magnitude of the current density is significantly stronger than that considered in a recent work by the same authors. We prove the existence of a solution which can be obtained by solving a non-convex minimization problem away from the boundary of the domain. Near the boundary, we show that this solution is essentially one-dimensional. We also establish some linear stability results for a simplified, one-dimensional version of the original problem.

1 Introduction

Superconducting materials are characterized by a complete loss of resistivity at temperatures below some critical threshold value. In this state, electrical current can flow through a superconducting sample while generating only a vanishingly small voltage drop. If the current is increased above a certain critical level, however, superconductivity is destroyed and the material reverts to the normal state—even while it remains below the critical temperature.

In this work, we study this phenomenon within the framework of the time-dependent Ginzburg-Landau model [1, 2], presented here in a dimensionless form

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} + i\phi u = (\nabla - iA)^2 u + u(1 - |u|^2) & \text{in } \Omega \times \mathbb{R}_+, \quad (1a) \\ -\kappa^2 \nabla \times \nabla \times A + \sigma \left(\frac{\partial A}{\partial t} + \nabla \phi \right) = \Im\{\bar{u} \nabla u\} + |u|^2 A & \text{in } \Omega \times \mathbb{R}_+, \quad (1b) \\ (i\nabla + A)u \cdot \nu = 0 \quad \text{and} \quad -\sigma \left(\frac{\partial A}{\partial t} + \nabla \phi \right) \cdot \nu = J & \text{on } \partial\Omega \times \mathbb{R}_+, \quad (1c) \\ u(x,0) = u_0 \quad \text{and} \quad A(x,0) = A_0 & \text{in } \Omega, \quad (1d) \end{array} \right. \quad (1e)$$

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In the above system of equations, u is the order parameter with $|u|$ representing the number density of superconducting electrons. Materials for which $|u| = 1$ are said to be purely superconducting while those for which $u = 0$ are said to be in the normal state. We denote the magnetic vector potential by A —so that the magnetic field is given by $h = \nabla \times A$ —and by ϕ the electric scalar potential. The constants κ and σ are the Ginzburg-Landau parameter and normal conductivity, of the superconducting material, respectively, and the quantity $-\sigma(A_t + \nabla\phi)$ is the normal current. All lengths in (1) have been scaled with respect to the coherence length ξ that characterizes spatial variations in u . The domain $\Omega \subset \mathbb{R}^2$ is smooth and the function $J : \partial\Omega \rightarrow \mathbb{R}$ represents the normal current entering the sample and hence must satisfy $\int_{\partial\Omega} J = 0$. . Note, that it is possible to prescribe the electric potential on $\partial\Omega$ instead of the current.

Except for the initial conditions, (1) is invariant under the gauge transformation [1]

$$A \rightarrow A + \nabla\omega \quad ; \quad u \rightarrow ue^{i\omega} \quad ; \quad \phi \rightarrow \phi - \frac{\partial\omega}{\partial t}$$

for some smooth ω . Finally, one has to prescribe h at a single point on $\partial\Omega$ for all $t > 0$ (cf. [3]).

It has been demonstrated in [3], for a fixed current, that in the limit $\kappa \rightarrow \infty$ one can formally obtain from (1) the following system of equations

$$\begin{cases} \frac{\partial u}{\partial t} + i\phi u = \Delta u + u \frac{1}{\epsilon^2} (1 - |u|^2), & \text{in } \Omega \times \mathbb{R}_+, & (2a) \\ \sigma \Delta \phi = \nabla \cdot [\Im(\bar{u} \nabla u)], & \text{in } \Omega \times \mathbb{R}_+, & (2b) \\ \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ and } -\sigma \frac{\partial \phi}{\partial \mathbf{n}} = J, & \text{on } \partial\Omega \times \mathbb{R}_+, & (2c) \\ u(x, 0) = u_0, & \text{in } \Omega, & (2d) \end{cases}$$

Similarly to (1), the system (2) remains invariant under the transformation

$$u \rightarrow e^{i\omega(t)} u \quad ; \quad \phi \rightarrow \phi - \frac{\partial\omega}{\partial t}.$$

This allows us to choose

$$\omega = \int_0^t \frac{(|u|^2 \phi)_\Omega(\tau)}{(|u|^2)_\Omega(\tau)} d\tau,$$

to guarantee that

$$(|u|^2 \phi)_\Omega(t) \equiv 0 \tag{3}$$

for all $t > 0$. Here

$$(f)_\Omega = \int_\Omega f \tag{4}$$

is the average of $f : \Omega \rightarrow \mathbb{R}$.

In the present contribution we consider steady-state solutions of (2). Let (u, ϕ) denote a smooth stationary solution of (2), which must therefore satisfy

$$\begin{cases} -\Delta u + i\phi u = \frac{u}{\epsilon^2} (1 - |u|^2), & \text{in } \Omega, \\ \sigma \Delta \phi = \nabla \cdot [\Im(\bar{u} \nabla u)], & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0, \quad -\sigma \frac{\partial \phi}{\partial \mathbf{n}} = J, \text{ and } \int_{\partial \Omega} J = 0, & \text{on } \partial \Omega. \end{cases} \quad \begin{matrix} (5a) \\ (5b) \\ (5c) \end{matrix}$$

If we set $u = \rho e^{i\chi}$ the problem assumes the form

$$\begin{cases} -\Delta \rho + \rho |\nabla \chi|^2 = \frac{\rho}{\epsilon^2} (1 - \rho^2), & \text{in } \Omega, \\ \operatorname{div}(\rho^2 \nabla \chi) = \rho^2 \phi, & \text{in } \Omega, \\ \sigma \Delta \phi = \operatorname{div}(\rho^2 \nabla \chi), & \text{in } \Omega, \\ \frac{\partial \rho}{\partial \mathbf{n}} = \frac{\partial \chi}{\partial \mathbf{n}} = 0, \quad -\sigma \frac{\partial \phi}{\partial \mathbf{n}} = J, \text{ and } \int_{\partial \Omega} J = 0, & \text{on } \partial \Omega. \end{cases} \quad \begin{matrix} (6a) \\ (6b) \\ (6c) \\ (6d) \end{matrix}$$

Note that (6) is invariant with respect to the transformation $\chi \rightarrow \chi + C$ for any constant C . We set $(\chi)_\Omega = 0$ in order to eliminate this degree of freedom throughout this paper.

The systems (2), (5), and (6) have attracted significant interest among both physicists [4]-[8] and mathematicians [3], [9]-[13] for a variety of domains and boundary conditions. A different simplification of (1) was derived by Du & Gray [14] for the same limit ($\kappa \rightarrow \infty$), but assuming that J and σ are of order $\mathcal{O}(\kappa^2)$ (cf. [15]). In [16] it has been established, when $\|J\|_\infty \ll 1/\epsilon$, that the above system possesses a stationary solution (u_s, ϕ_s) satisfying

$$\|1 - |u_s|\|_{2,2} \leq C\epsilon^2 \|J\|_\infty^2.$$

Furthermore, it is shown in [16] that (u_s, ϕ_s) is linearly stable.

In the present paper we focus on steady state solutions of (2) for $\|J\| \sim \mathcal{O}(1/\epsilon)$. Far away from the boundaries we attempt to approximate the solution (u_s, ϕ_s) by $\phi_s \approx 0$ and $|u_s|^2 \approx 1 - |\nabla \zeta|^2$ and $\chi_s \approx \zeta/\epsilon$, where ζ is given as the solution of

$$\begin{cases} \operatorname{div}([1 - |\nabla \zeta|^2] \nabla \zeta) = 0, & \text{in } \Omega, \\ [1 - |\nabla \zeta|^2] \frac{\partial \zeta}{\partial \nu} = j, & \text{on } \partial \Omega, \end{cases} \quad (7)$$

in which $j = \epsilon J$. To this end, let

$$\mathcal{W}_{3,\alpha} := \left\{ u \in C^{3,\alpha}(\bar{\Omega}) \mid \int_{\Omega} u \, dx = 0 \right\}.$$

Then the following existence/uniqueness result holds for the system (7).

Proposition 1. *For a sufficiently small $\mu > 0$, there exists a solution $\zeta_\mu \in \mathcal{W}_{3,\alpha}$ of (6) with $j = \mu j_r$, for $j_r \in C^{3,\alpha}(\partial \Omega)$. Furthermore, the solution exists for all sufficiently small μ for which*

$$\|\nabla \zeta_\mu\|_{L^\infty(\partial \Omega)} < \frac{1}{\sqrt{3}}. \quad (8)$$

Finally, the above solution is unique as long as (8) is satisfied.

Once the existence of the above “outer” solution is established, we establish the existence of a boundary layer approximation. Let $t = d(x, \partial\Omega)$ and $\tau = t/\epsilon$. Set further $\sigma_0 = \sigma/\epsilon^2$, $\rho_i = |u|$, $\rho_r = [1 - |\nabla\zeta \times \nu|^2]^{1/2}|_{t=0}$ and $\varphi_i = \epsilon^2\phi$. The boundary layer approximation is given by the solution of

$$\begin{cases} -\rho_i'' - \left(\rho_r^2 - \frac{(\sigma_0\varphi_i' - j)^2}{\rho_i^4} - \rho_i^2\right)\rho_i = 0 & \text{in } \mathbb{R}_+, \\ -\sigma_0\varphi_i'' + \rho_i^2\varphi_i = 0 & \text{in } \mathbb{R}_+, \\ \rho_i'(0) = 0, \\ \varphi_i'(0) = \frac{j}{\sigma_0}, \end{cases} \quad (9)$$

where $' = d/d\tau$. We prove the following lemma.

Lemma 1.1. *Suppose that (8) holds. For a sufficiently large σ_0 , there exists a unique solution (ρ_i, φ_i) of (9) such that $\rho_i \geq [1 - |\nabla\zeta|^2]^{1/2}(t=0)$ and $(\rho_i - 1 + |\nabla\zeta|^2, \varphi_i) \in H^1(\mathbb{R}_+, \mathbb{R}^2)$. Furthermore, for some positive γ and C , we have*

$$|\varphi_i| + |\varphi_i'| + |\rho_i^2 - 1 + |\nabla\zeta(t=0)|^2|^{1/2} + |\rho_i'| < Ce^{-\gamma\sigma_0^{-1/2}\tau}.$$

We also demonstrate that, when $\sigma_0 \rightarrow \infty$, a good approximation for (ρ_i, φ_i) is given by the solution of

$$\begin{cases} \rho_r^2 - \frac{(\sigma_0\varphi_i' - j)^2}{\rho_i^4} - \rho_i^2 = 0 & \text{in } \mathbb{R}_+, \\ -\sigma_0\varphi_i'' + \rho_i^2\varphi_i = 0 & \text{in } \mathbb{R}_+, \\ \rho_i'(0) = 0, \\ \varphi_i'(0) = \frac{j}{\sigma_0}. \end{cases}$$

Finally, we combine the solutions of (6) and (12) to obtain a uniform approximation of (ρ, χ, ϕ) , denoted by (ρ_0, χ_0, ϕ_0) (whose precise definition is given by (110)) and satisfies $\rho_0 \approx [1 - |\nabla\zeta|^2]^{1/2}$ for $d(x, \partial\Omega) \gg \epsilon$ and $\rho_0 \approx \rho_{i0}$ when $d(x, \partial\Omega) \sim \mathcal{O}(\epsilon)$. Then, we establish existence of a solution for the system (2) satisfying

Theorem 1. *For some $\epsilon_0 > 0$, there exists, for all $\epsilon < \epsilon_0$, a solution of (13) satisfying*

$$\|\rho - \rho_0\|_2 + \epsilon(\|\chi - \chi_0\|_2 + \|\phi - \phi_0\|_2) \leq C\epsilon^{2+s/2}, \quad (10)$$

and

$$\|\rho - \rho_0\|_\infty + \|\rho - \rho_0\|_{1,2} + \epsilon(\|\chi - \chi_0\|_{2,2} + \|\phi - \phi_0\|_{2,2}) \leq C\epsilon^{1+s/2}, \quad (11)$$

for all $s < 1$.

The rest of the paper is organized as follows. In the next section we consider an outer approximation of solutions of (6). In Section 3, we consider a matching inner solution, while Section 4 is devoted to the development of a uniform approximation and the proof of Theorem 1. Finally, in the last section, we establish some linear stability results for a corresponding simplified, one-dimensional problem.

2 Outer approximation

Suppose that $\|J\|\epsilon \sim \mathcal{O}(1)$ and $\sigma = \sigma_0\epsilon^2$ for some $\sigma_0 > 0$. The solution of the problem (6) will be obtained by "gluing" outer and inner approximations, constructed away from and near the boundary, respectively.

Combining (6b-c), we note that ϕ satisfies

$$\sigma_0\epsilon^2\Delta\phi = \rho^2\phi.$$

We assume that the outer solution—outside of a $\mathcal{O}(\epsilon)$ -thin inner layer near $\partial\Omega$ —corresponds to a superconducting state in which the magnitude ρ of the order parameter is bounded away from zero. By following the standard argument, this implies that ϕ is exponentially small in the outer region so that we can set $\phi_{out} \equiv 0$ away from the boundary. This observation leads to the approximate problem

$$\begin{cases} -\Delta\rho_{out} + \rho_{out}|\nabla\chi_{out}|^2 = \frac{\rho_{out}}{\epsilon^2}(1 - \rho_{out}^2), & \text{in } \Omega, \\ \operatorname{div}(\rho_{out}^2\nabla\chi_{out}) = 0, & \text{in } \Omega, \\ \frac{\partial\rho_{out}}{\partial\mathbf{n}} = 0, \quad \rho_{out}^2\frac{\partial\chi_{out}}{\partial\mathbf{n}} = -J, \text{ and } \int_{\partial\Omega} J = 0, & \text{on } \partial\Omega. \end{cases} \quad (12)$$

Here the boundary condition reflects the conjecture that the normal current $-\sigma\nabla\phi$ turns into a superconducting one $\rho_{out}^2\nabla\chi_{out}$ within a thin one-dimensional boundary layer. We seek an approximation to the solution of (12) in the limit $\epsilon \rightarrow 0$. Since $J \sim \mathcal{O}(\frac{1}{\epsilon})$ and $\rho_{out} \sim \mathcal{O}(1)$ is bounded away from 0, we have that $\nabla\chi_{out} \sim \mathcal{O}(1/\epsilon)$. Using the first equation in (12), we obtain that to leading order

$$|\nabla\chi_{out,0}|^2 = \frac{1}{\epsilon^2}(1 - \rho_{out,0}^2). \quad (13)$$

It follows that

$$\begin{cases} \operatorname{div}([1 - \epsilon^2|\nabla\chi_{out,0}|^2]\nabla\chi_{out,0}) = 0, & \text{in } \Omega, \\ [1 - \epsilon^2|\nabla\chi_{out,0}|^2]\frac{\partial\chi_{out,0}}{\partial\mathbf{n}} = -J \text{ and } \int_{\partial\Omega} J = 0, & \text{on } \partial\Omega. \end{cases} \quad (14)$$

Rescaling

$$\zeta = \epsilon\chi_{out,0}, \quad j = -\epsilon J,$$

yields

$$\begin{cases} \operatorname{div}([1 - |\nabla\zeta|^2]\nabla\zeta) = 0, & \text{in } \Omega, \\ [1 - |\nabla\zeta|^2]\frac{\partial\zeta}{\partial\mathbf{n}} = j \text{ and } \int_{\partial\Omega} j = 0, & \text{on } \partial\Omega. \end{cases} \quad (15)$$

We seek smooth (at least, $H^3(\Omega)$) solutions of (15) satisfying

$$\|\nabla\zeta\|_\infty \leq 1. \quad (16)$$

The latter inequality is needed to guarantee that $\rho_{out,0}$ in (13) remains meaningful. In fact, as will be discussed below, a stronger bound on $\|\nabla\zeta\|_\infty$ will be required to establish existence of solutions of (15).

First, we observe that the inequality constraint (16) and the elementary statement

$$\max_{t \in [0,1]} (t - t^3) = \frac{2}{3\sqrt{3}},$$

result in the bound

$$\|\mathbf{j}_s\|_\infty \leq \frac{2}{3\sqrt{3}}, \quad (17)$$

on the superconducting current $\mathbf{j}_s := [1 - |\nabla\zeta|^2] \nabla\zeta$. A necessary condition for existence of solutions of (15) then follows from the requirement that the boundary data should satisfy (17), that is $\|j\|_{L^\infty(\partial\Omega)} \leq \frac{2}{3\sqrt{3}}$. In fact, an even stronger necessary condition can be established as we will demonstrate in the next proposition.

Let the distance between any two points $x, y \in \Omega$ be defined as

$$d(x, y) = \inf_{\gamma \subset \bar{\Omega}} \mathbf{L}(\gamma),$$

where γ is a continuous path connecting x and y and $\mathbf{L}(\gamma)$ is the length of γ . If Ω is convex, then $d(x, y) = |x - y|$. For any $x, y \in \partial\Omega$, let

$$M(x, y) = \frac{1}{d(x, y)} \left| \int_\Gamma j \, ds \right|,$$

where Γ is either of the two paths in $\partial\Omega$ connecting x and y (note that the value of M is independent of the choice of the path due to the condition $\int_{\partial\Omega} j = 0$).

Proposition 2. *Suppose that the boundary value problem (15) has a solution in $H^3(\Omega)$. Then*

$$\sup_{x, y \in \partial\Omega} M(x, y) \leq \frac{2}{3\sqrt{3}}. \quad (18)$$

Proof. Given $x, y \in \partial\Omega$, let $\tilde{\Gamma}$ denote a shortest path in $\bar{\Omega}$ connecting y to x . Suppose that a solution $\zeta \in H^3(\Omega)$ of (15) exists. Clearly,

$$\int_\Gamma j \, ds + \int_{\tilde{\Gamma}} [1 - |\nabla\zeta|^2] \frac{\partial\zeta}{\partial\mathbf{n}} \, ds = 0 \quad (19)$$

and the inequality

$$\left| \int_{\tilde{\Gamma}} [1 - |\nabla\zeta|^2] \frac{\partial\zeta}{\partial\mathbf{n}} \, ds \right| \leq \|\mathbf{j}_s\|_\infty d(x, y) \leq \frac{2}{3\sqrt{3}} d(x, y)$$

holds by (17). Substituting this expression into (19) yields

$$M(x, y) \leq \frac{2}{3\sqrt{3}}$$

and (18) follows because $x, y \in \partial\Omega$ were chosen arbitrarily. \square

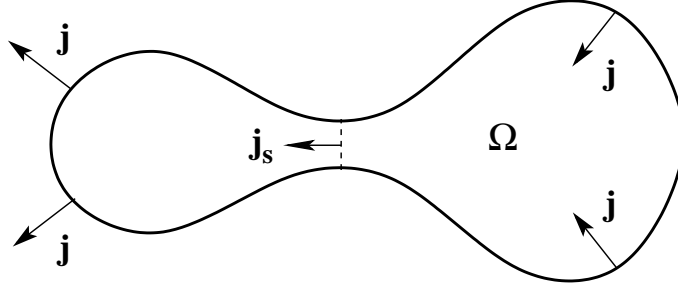


Figure 1: Global constraint on the boundary data—the superconducting current through the narrowing in Ω has to satisfy the pointwise bound $|\mathbf{j}_s| \leq \frac{2}{3\sqrt{3}}$.

Remark 2.1. It can be easily verified that a solution $\zeta = jx$ of (15) obtained in [4] when $\Omega = \mathbb{R}$, exists only if the current at infinity satisfies $|j| \leq 2/(3\sqrt{3})$. Proposition 2 establishes that the geometry of the domain Ω places additional restrictions on the boundary data in (15) when $\Omega \subset \mathbb{R}^2$, as illustrated in Figure 1.

In order to prove existence of solutions to (15) we will need the following auxiliary lemma

Lemma 2.1. *Let ζ denote a $C^3(\Omega)$ solution of (15). If $\|\nabla\zeta\|_\infty < 1/\sqrt{3}$, then $\|\nabla\zeta\|_\infty = \|\nabla\zeta\|_{L^\infty(\partial\Omega)}$.*

Proof. By expanding the left hand side of (15) we obtain

$$(1 - |\nabla\zeta|^2) \Delta\zeta - \nabla(|\nabla\zeta|^2) \cdot \nabla\zeta = 0,$$

then taking the gradient of both sides in this expression, in turn, gives

$$(1 - |\nabla\zeta|^2) \nabla\Delta\zeta - \Delta\zeta \nabla(|\nabla\zeta|^2) - \nabla\nabla(|\nabla\zeta|^2) \nabla\zeta - \nabla\nabla\zeta \nabla(|\nabla\zeta|^2) = 0.$$

Solving for $\nabla\Delta\zeta$ and taking dot product with $\nabla\zeta$, we have

$$\begin{aligned} \nabla\zeta \cdot \nabla\Delta\zeta &= \frac{1}{(1 - |\nabla\zeta|^2)} (\Delta\zeta \nabla\zeta \cdot \nabla(|\nabla\zeta|^2) \\ &\quad + \frac{1}{2} |\nabla(|\nabla\zeta|^2)|^2 + \nabla\zeta \cdot \nabla\nabla(|\nabla\zeta|^2) \nabla\zeta). \end{aligned} \quad (20)$$

Now let

$$\mathcal{L} = A(\nabla\zeta) : \nabla\nabla - b \cdot \nabla,$$

where $A : \mathbb{R}^2 \rightarrow M^{2 \times 2}$ and $b \in C^1(\Omega, \mathbb{R}^2)$ are given by

$$A(z) = (1 - |z|^2) \mathbf{I} - 2z \otimes z \quad (21)$$

and

$$b = \nabla(|\nabla\zeta|^2) + 2\Delta\zeta \nabla\zeta,$$

respectively. Observing that

$$\Delta(|\nabla u|^2) = 2|\nabla \nabla u|^2 + 2\nabla u \cdot \nabla \Delta u, \quad (22)$$

holds for any $u \in C^3(\Omega)$, substituting (20) into (22), and rearranging terms we find that

$$\mathcal{L}|\nabla \zeta|^2 = 2(1 - |\nabla \zeta|^2)|\nabla \zeta|^2 \geq 0.$$

The operator \mathcal{L} is uniformly elliptic when $|\nabla \zeta| < 1/\sqrt{3}$. Indeed, if we let $\nabla \zeta^\perp := (-\zeta_y, \zeta_x)$, it immediately follows from (21) that $\nabla \zeta^\perp$ and $\nabla \zeta$ are eigenvectors of $A(\nabla \zeta)$ with eigenvalues $1 - |\nabla \zeta|^2$ and $1 - 3|\nabla \zeta|^2$, respectively. Both eigenvalues are positive and the matrix $A(\nabla \zeta)$ is positive definite as long as $|\nabla \zeta| < 1/\sqrt{3}$. The proof of the lemma follows by the maximum principle. \square

We now state the first existence and uniqueness result for (15). Given $0 < \alpha < 1$ and $k \in \mathbb{N}$, let $j_r \in C^{k-1,\alpha}(\partial\Omega)$ satisfy $\int_{\partial\Omega} j_r = 0$ and $\|j_r\|_\infty = 1$. Set

$$\mathcal{W}_{k,\alpha} := \left\{ u \in C^{k,\alpha}(\bar{\Omega}) \mid \int_{\Omega} u \, dx = 0 \right\}.$$

For convenience, we repeat here the statement of Proposition 1

Proposition 3. *For a sufficiently small $\mu > 0$, there exists a solution $\zeta_\mu \in \mathcal{W}_{k,\alpha}$ of (15) with $j = \mu j_r$. Furthermore, the solution exists for all sufficiently small μ that satisfy*

$$\|\nabla \zeta_\mu\|_{L^\infty(\partial\Omega)} < \frac{1}{\sqrt{3}}. \quad (23)$$

Finally, the above solution is unique as long as (23) is satisfied.

Proof. We use the implicit function theorem. Define $F : \mathcal{W}_{k,\alpha} \times \mathbb{R} \rightarrow Z$ by

$$F(u, \mu) = \left(-\operatorname{div}([1 - |\nabla u|^2] \nabla u), [1 - |\nabla u|^2] \nabla u \cdot \mathbf{n} - \mu j_r \right),$$

where

$$Z = \{(z_1, z_2) \in C^{k-2,\alpha}(\Omega) \times C^{k-1,\alpha}(\partial\Omega) ; \int_{\Omega} z_1 + \int_{\partial\Omega} z_2 = 0\}.$$

Clearly, $F(0, 0) = 0$. Furthermore, it can be readily verified that the Frechet derivative of F with respect to u is given by

$$D_u F(u, \mu)\omega = \left(-\operatorname{div}(A(\nabla u)\nabla \omega), A(\nabla u)\nabla \omega \cdot \mathbf{n} \right),$$

where A is as defined in (21). In particular, by standard elliptic estimates,

$$D_u F(0, 0) = \left(-\Delta \omega, \frac{\partial \omega}{\partial \mathbf{n}} \right),$$

is an isomorphism of $\mathcal{W}_{k,\alpha}$ onto Z . It follows that a solution of (15) with $j = \mu j_r$ exists in some neighborhood of $\mu = 0$. We denote this solution by ζ_μ .

By the implicit function theorem, the solution continues to exist as long as $D_u F(\zeta_\mu, \mu)$ is an isomorphism from $\mathcal{W}_{k,\alpha}$ onto Z . To show that this is indeed the case—as long as (23) holds—we first notice that the map $D_u F(\zeta_\mu, \mu)$ is injective. Indeed, if $D_u F(\zeta_\mu, \mu)\omega = 0$ for some $\omega \in \mathcal{W}_{k,\alpha}$, i.e.,

$$\begin{cases} -\operatorname{div}(A(\nabla \zeta_\mu) \nabla \omega) \omega = 0 & \text{in } \Omega, \\ A(\nabla \zeta_\mu) \nabla \omega \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (24)$$

then multiplying the equation in (24) by ω and using Green formula yields,

$$0 = \int_{\Omega} A(\nabla \zeta_\mu) \nabla \omega \cdot \nabla \omega \, dx \geq \|1 - 3|\nabla \zeta_\mu|^2\|_{\infty} \|\nabla \omega\|_2,$$

implying that $\omega = 0$ (under the assumption (23)). To prove that $D_u F(\zeta_\mu, \mu)$ is onto Z , consider any $(z_1, z_2) \in Z$ and the problem,

$$\begin{cases} -\operatorname{div}(A(\nabla \zeta_\mu) \nabla \omega) \omega = z_1 & \text{in } \Omega, \\ A(\nabla \zeta_\mu) \nabla \omega \cdot \mathbf{n} = z_2 & \text{on } \partial\Omega. \end{cases} \quad (25)$$

Existence for (25) can be established by using the Schauder approach along with the Fredholm alternative (see Theorem 6.30, Theorem 6.31 and the subsequent remark in [17]). Alternatively, one can prove existence of a weak solution $\omega \in H^1(\Omega)$ to (25) by minimization of the functional

$$J(u) = \int_{\Omega} \left[A(\nabla \zeta_\mu) \nabla u \cdot \nabla u - z_1 u \right] - \int_{\partial\Omega} z_2 u \quad \text{over } u \in H^1(\Omega) \text{ with } \int_{\Omega} u = 0,$$

and then deduce further regularity for ω by standard elliptic estimates (see [18]).

Once again, it follows by the implicit function theorem that as long as $\|\nabla \zeta_\mu\|_{\infty} < 1/\sqrt{3}$ the branch (ζ_μ, μ) continuously extends from $(0, 0)$. By Lemma 2.1 the maximum of $|\nabla \zeta_\mu|$ is attained on the boundary.

To complete the proof we need to demonstrate uniqueness of the above solutions for every fixed μ . To this end we define the functional

$$I(\zeta) = - \int_{\Omega} \left[\frac{1}{4} (1 - |\nabla \zeta|^2)^2 + \nabla_{\perp} \Phi \cdot \nabla \zeta \right] dx, \quad (26)$$

where Φ is any C^1 potential, whose tangential derivative along $\partial\Omega$ satisfies $\partial\Phi/\partial\tau = \mu j_r$. It can be readily verified that every critical point of I must satisfy (15). Furthermore, I is convex in

$$X = \left\{ \zeta \in H^1(\Omega) \mid \|\nabla \zeta\|_{\infty} \leq 1/\sqrt{3}; \int_{\Omega} \zeta \, dx = 0 \right\}.$$

and it is strictly convex in the interior of X . Thus, for every μ , the functional I can have at most one critical point, which must therefore lie on the branch of solutions ζ_μ whose existence has been established above. \square

Next order term: Suppose that there exists a solution for (15) satisfying

$$\|\nabla\zeta\|_\infty < \frac{1}{\sqrt{3}}. \quad (27)$$

Next we seek a more accurate estimate of the solution of (12) when $\epsilon \ll 1$. To this end, we set

$$(\rho_{out}, \chi_{out}) = (\rho_{out,0}, \chi_{out,0}) + \epsilon^2(\rho_{out,1}, \chi_{out,1}) + \dots,$$

where $\rho_{out,0}$ and $\chi_{out,0}$ solve (13)-(14). The next order balances for ρ and χ then take the form

$$\begin{cases} \rho_{out,1}(1 - 3\rho_{out,0}^2 - \epsilon^2|\nabla\chi_{out,0}|^2) - 2\epsilon^2\rho_{out,0}\nabla\chi_{out,0} \cdot \nabla\chi_{out,1} = -\Delta\rho_{out,0}, \\ \operatorname{div}(\rho_{out,0}^2\nabla\chi_{out,1} + 2\rho_{out,0}\rho_{out,1}\nabla\chi_{out,0}) = 0, \end{cases} \quad (28)$$

respectively. Recalling that $\rho_{out,0}^2 = 1 - |\nabla\zeta|^2 > 2/3$ in Ω and using (13), we obtain

$$\rho_{out,1} = \frac{\Delta\rho_{out,0}}{2\rho_{out,0}^2} - \frac{\epsilon^2\nabla\chi_{out,0} \cdot \nabla\chi_{out,1}}{\rho_{out,0}}.$$

Substituting this expression into the second equation in (28) gives the following problem

$$\begin{cases} \operatorname{div}(A(\nabla\zeta)\nabla\zeta_1) = -\operatorname{div}\left(\frac{\Delta\rho_{out,0}}{\rho_{out,0}}\nabla\zeta\right) & \text{in } \Omega, \\ A(\nabla\zeta)\nabla\zeta_1 \cdot \mathbf{n} = -\frac{\Delta\rho_{out,0}}{\rho_{out,0}}\nabla\zeta \cdot \mathbf{n} & \text{on } \partial\Omega, \end{cases} \quad (29)$$

where A is given by (21) and

$$\zeta_1 = \epsilon\chi_{out,1}.$$

Since $A(\nabla\zeta) \in C^{k-1,\alpha}(\Omega, M(2,2))$ is positive definite by (27) and the right-hand-side of (29) is in $C^{k-4,\alpha}(\Omega)$, we may use Schauder estimates (cf. for instance Theorem 9.3 and the subsequent Remark 2 in [18]) to conclude the existence of a unique $C^{k-2,\alpha}$ -solution of (29). Furthermore, we have that

$$\|\zeta_1\|_{k-2,\alpha} \leq C(\Omega, \alpha, k). \quad (30)$$

Next, we set

$$\rho_o = \rho_{out,0} + \epsilon^2\rho_{out,1} \quad ; \quad \chi_o = \chi_{out,0} + \epsilon^2\chi_{out,1}. \quad (31)$$

With the aid of (13) and (28) we then obtain that

$$-\Delta\rho_o - \frac{1}{\epsilon^2}\rho_o(1 - \rho_o^2 - \epsilon^2|\nabla\chi_o|^2) = g_1 \quad (32)$$

where

$$g_1 = \epsilon^2 \left[-\Delta\rho_{out,1} + \rho_{out,0}(\rho_{out,1}^2 + |\nabla\zeta_1|^2) + \rho_{out,1}(2\rho_{out,0}\rho_{out,1} + 2\nabla\zeta \cdot \nabla\zeta_1 + \epsilon^2(\rho_{out,1}^2 + |\nabla\zeta_1|^2)) \right].$$

If we choose $k \geq 4$, then (30) implies that

$$\|g_1\|_\infty \leq C\epsilon^2, \quad (33)$$

for some $C > 0$. In a similar manner, it also follows that

$$\operatorname{div}(\rho_o^2 \nabla \chi_o) = g_2, \quad (34)$$

where

$$\|g_2\|_\infty \leq C\epsilon^2. \quad (35)$$

3 Inner solution

Next, the outer solution has to be "bridged" to the boundary conditions on $\partial\Omega$ in (6) by constructing an appropriate inner solution near the boundary. We thus consider (6) in a $\mathcal{O}(\epsilon)$ -thick boundary layer near $\partial\Omega$. We begin by introducing a curvilinear coordinate system (s, t) near $\partial\Omega$ by setting

$$(x, y) = \mathbf{r}(s) - t \mathbf{n}(s)$$

where $t = d(x, \partial\Omega)$ is the distance function to $\partial\Omega$ that we will assume to be positive in the interior of Ω . The vector function \mathbf{r} describes $\partial\Omega$ and is parametrized with respect to the arclength s calculated from some fixed initial point on $\partial\Omega$ in the counterclockwise direction. The outward unit normal vector $\mathbf{n}(s)$ to $\partial\Omega$ at $\mathbf{r}(s)$ is given by $\mathbf{r}_{ss}(s) = -\kappa(s)\mathbf{n}(s)$, where $\kappa(s)$ denotes the curvature of $\partial\Omega$ at the point $\mathbf{r}(s)$. The Jacobian of the transformation $(x, y) \rightarrow (s, t)$ is then

$$\mathbf{g} = 1 - t\kappa(s). \quad (36)$$

Recall that $\sigma = \sigma_0\epsilon^2$. Then after rescaling

$$\tau = \frac{t}{\epsilon} \quad ; \quad j = -\epsilon J \quad ; \quad \varphi = \epsilon^2 \phi, \quad (37)$$

and rewriting the system (6) in terms of (s, t) , we have

$$\left\{ \begin{array}{ll} -\frac{\partial^2 \rho}{\partial \tau^2} - \left(1 - \left|\frac{\partial v}{\partial \tau} + \frac{\partial \zeta}{\partial t}(s, \epsilon\tau)\right|^2 - \rho^2\right)\rho = \\ \quad -\epsilon \frac{\kappa}{\mathbf{g}} \frac{\partial \rho}{\partial \tau} + \epsilon^2 \left(\frac{1}{\mathbf{g}} \frac{\partial}{\partial s}\right)^2 \rho - \epsilon^2 \left|\frac{1}{\mathbf{g}} \left(\frac{\partial v}{\partial s} + \frac{1}{\epsilon} \frac{\partial \zeta}{\partial s}(s, \epsilon\tau)\right)\right|^2 \rho & \text{in } \Omega \quad (38a) \\ -\sigma_0 \frac{\partial^2 \varphi}{\partial \tau^2} + \rho^2 \varphi = -\sigma_0 \epsilon \frac{\kappa}{\mathbf{g}} \frac{\partial \varphi}{\partial \tau} + \sigma_0 \epsilon^2 \left(\frac{1}{\mathbf{g}} \frac{\partial}{\partial s}\right)^2 \varphi & \text{in } \Omega \quad (38b) \\ \frac{\partial}{\partial \tau} \left(\rho^2 \left(\frac{\partial v}{\partial \tau} + \frac{\partial \zeta}{\partial t}(s, \epsilon\tau) \right) \right) - \sigma_0 \frac{\partial^2 \varphi}{\partial \tau^2} = -\sigma_0 \epsilon \frac{\kappa}{\mathbf{g}} \frac{\partial \varphi}{\partial \tau} + \sigma_0 \epsilon^2 \left(\frac{1}{\mathbf{g}} \frac{\partial}{\partial s}\right)^2 \varphi \\ \quad + \epsilon \rho^2 \frac{\kappa}{\mathbf{g}} \left(\frac{\partial v}{\partial \tau} + \frac{\partial \zeta}{\partial t}(s, \epsilon\tau) \right) - \epsilon^2 \frac{1}{\mathbf{g}} \frac{\partial}{\partial s} \left(\rho^2 \frac{1}{\mathbf{g}} \left(\frac{\partial v}{\partial s} + \frac{1}{\epsilon} \frac{\partial \zeta}{\partial s}(s, \epsilon\tau) \right) \right) & \text{in } \Omega \quad (38c) \\ \frac{\partial \rho}{\partial \tau} = 0, \quad \frac{\partial v}{\partial \tau} = \frac{\partial \zeta}{\partial \mathbf{n}}, \quad \frac{\partial \varphi}{\partial \tau} = \frac{j}{\sigma_0}, \quad \text{and} \quad \int_{\partial\Omega} j = 0 & \text{on } \partial\Omega \quad (38d) \\ \int_{\Omega} v \, dx = 0, & \quad (38e) \end{array} \right.$$

where

$$v(s, \tau) = \chi(s, \tau) - \frac{1}{\epsilon} \zeta(s, \epsilon \tau), \quad (39)$$

and (38b) is obtained by combining (6b) and (6c). Note that the second boundary condition in (38d) can be written as

$$\frac{\partial v}{\partial \tau} = \frac{j}{1 - |\nabla \zeta|^2} \text{ on } \partial \Omega,$$

by taking into account (15).

We will attempt to obtain the inner solution for (38) through a one-dimensional approximation in terms of the variable τ in the direction transverse to $\partial \Omega$. To this end, let $(\rho_{i0}, \varphi_{i0}, v_{i0})$ denote the solution of the following problem

$$\begin{cases} -\rho_{i0}'' - \left(\rho_r^2(s) - \left| v_{i0}' + \frac{\partial \zeta}{\partial t}(s, 0) \right|^2 - \rho_{i0}^2 \right) \rho_{i0} = 0 & \text{in } \mathbb{R}_+ \end{cases} \quad (40a)$$

$$\begin{cases} -\sigma_0 \varphi_{i0}'' + \rho_{i0}^2 \varphi_{i0} = 0 & \text{in } \mathbb{R}_+ \end{cases} \quad (40b)$$

$$\begin{cases} \left(\rho_{i0}^2 \left(v_{i0}' + \frac{\partial \zeta}{\partial t}(s, 0) \right) \right)' - \rho_{i0}^2 \varphi_{i0} = 0 & \text{in } \mathbb{R}_+ \end{cases} \quad (40c)$$

$$\begin{cases} \rho_{i0}'(0) = 0 \end{cases} \quad (40d)$$

$$\begin{cases} \varphi_{i0}'(0) = \frac{j(s)}{\sigma_0} \end{cases} \quad (40e)$$

$$\begin{cases} v_{i0}'(0) = -\frac{\partial \zeta}{\partial t}(s, 0), \end{cases} \quad (40f)$$

where

$$\rho_r^2(s) = 1 - \left| \frac{\partial \zeta}{\partial s}(s, 0) \right|^2.$$

In what follows, we drop the dependence on s for notational simplicity.

Adding the second and the third equations in (40) and integrating, we find that v_{i0} can be determined up to a constant by solving

$$v_{i0}' = \frac{\sigma_0 \varphi_{i0}' - j}{\rho_{i0}^2} - \frac{\partial \zeta}{\partial t}(s, 0). \quad (41)$$

Then $(\rho_{i0}, \varphi_{i0})$ satisfy

$$\begin{cases} -\rho_{i0}'' - \left(\rho_r^2 - \frac{(\sigma_0 \varphi_{i0}' - j)^2}{\rho_{i0}^4} - \rho_{i0}^2 \right) \rho_{i0} = 0 & \text{in } \mathbb{R}_+ \\ -\sigma_0 \varphi_{i0}'' + \rho_{i0}^2 \varphi_{i0} = 0 & \text{in } \mathbb{R}_+ \\ \rho_{i0}'(0) = 0 \\ \varphi_{i0}'(0) = \frac{j}{\sigma_0}. \end{cases} \quad (42)$$

As $\tau \rightarrow \infty$ we expect that $(\varphi_{i0}', \rho_{i0}') \rightarrow (0, 0)$ and hence $\lim_{\tau \rightarrow \infty} \rho_{i0} = \rho_j$, where ρ_j solves

$$\rho_r^2 - \frac{j^2}{\rho_j^4} - \rho_j^2 = 0. \quad (43)$$

It is an easy exercise to show that positive solutions of (43) exist as long as

$$j^2 \leq \frac{4}{27}\rho_r^6. \quad (44)$$

On the other hand, since $j^2 = (1 - |\nabla\zeta|^2)^2 |\partial\zeta/\partial\mathbf{n}|^2$ on $\partial\Omega$, it follows that

$$\rho_j^2 = (1 - |\nabla\zeta|^2)|_{\partial\Omega}. \quad (45)$$

By (15) and (27) we have the following relationship

$$j^2 = (1 - |\nabla\zeta|^2)^2 \left| \frac{\partial\zeta}{\partial\mathbf{n}} \right|^2 = -(1 - |\nabla\zeta|^2)^3 + \rho_r^2 (1 - |\nabla\zeta|^2)^2 < \frac{4}{9} \left(\rho_r^2 - \frac{2}{3} \right) \leq \frac{4}{27} \rho_r^6,$$

on $\partial\Omega$. Here the two sides are equal only when $\rho_r = 1$, therefore the inequality constraint (27) is stronger than that in (44).

We repeat again, for the convenience of the reader, the statement of Lemma 1.1.

Lemma 3.1. *Suppose that the strict inequality in the condition (44) holds. For a sufficiently large σ_0 , there exists a unique solution $(\rho_{i0}, \varphi_{i0})$ of (42) such that $\rho_{i0} \geq \rho_j$ and $(\rho_{i0} - 1 + |\nabla\zeta(s, 0)|^2, \varphi_{i0}) \in H^1(\mathbb{R}_+, \mathbb{R}^2)$. Furthermore, for some positive γ and C , we have*

$$|\varphi_{i0}| + |\varphi'_{i0}| + |\rho_{i0}^2 - 1 + |\nabla\zeta(s, 0)|^2|^{1/2} + |\rho'_{i0}| + |v'_{i0}| < Ce^{-\gamma\sigma_0^{-1/2}\tau} \quad (46)$$

Proof. Note that, given a sufficiently large σ_0 and $\rho_r = 1$, the existence of a solution subject to the Dirichlet boundary condition $\rho_{i0}(0) = 0$ was proved in [3] for a problem similar to (42). Here we will adopt a different strategy relying on a large σ_0 expansion argument that was formally presented in [3].

Without any loss of generality we may assume that $j \leq 0$, because we can apply the transformation $(j, \varphi_{i0}) \rightarrow (-j, -\varphi_{i0})$ otherwise. Following the rescaling

$$\eta = \rho_r \frac{\tau}{\sigma_0^{1/2}}; \quad \vartheta(\eta) = \sigma_0^{1/2} \frac{\varphi_{i0}(\sigma_0^{1/2} \rho_r^{-1} \eta)}{\rho_r^2}; \quad \mu(\eta) = \frac{\rho_{i0}(\sigma_0^{1/2} \rho_r^{-1} \eta)}{\rho_r}; \quad j_r = \frac{j}{\rho_r^3}, \quad (47)$$

the problem (42) takes the form

$$\begin{cases} -\frac{1}{\sigma_0} \mu'' - \left(1 - \frac{(\vartheta' - j_r)^2}{\mu^4} - \mu^2 \right) \mu = 0 & \text{in } \mathbb{R}_+, \\ -\vartheta'' + \mu^2 \vartheta = 0 & \text{in } \mathbb{R}_+, \\ \mu'(0) = 0, \\ \vartheta'(0) = j_r. \end{cases} \quad (48)$$

The leading order approximation (μ_0, ϑ_0) for (48) in σ_0^{-1} is obtained by neglecting the $\mathcal{O}(\sigma_0^{-1})$ -terms in (48). Thus we look for the solution of

$$\begin{cases} \mu_0^4(1 - \mu_0^2) = (\vartheta_0' - j_r)^2 & \text{in } \mathbb{R}_+, \\ -\vartheta_0'' + \mu_0^2 \vartheta_0 = 0 & \text{in } \mathbb{R}_+, \\ \vartheta_0'(0) = j_r. \end{cases} \quad (49)$$

Because

$$j_r^2 < \frac{4}{27}$$

due to the assumption that the strict inequality holds in (44), the algebraic equation for μ_0^2 has two distinct positive solutions whenever $j_r \leq \vartheta'_0 \leq 0$. We choose the solution for which

$$\frac{2}{3} < \mu_0^2 \leq 1. \quad (50)$$

Step 1: Prove the existence of a solution (μ_0, ϑ_0) of (49), where $(\mu_0 - \mu_j, \vartheta_0) \in H^1(\mathbb{R}_+, \mathbb{R}^2)$ and $\mu_j \in \left(\sqrt{2/3}, 1\right]$ solves

$$\mu_j^4(1 - \mu_j^2) = j_r^2. \quad (51)$$

We can reduce the system (49) to a single equation for ϑ_0 and then obtain μ_0 from $\vartheta'_0 - j_r$ by solving an algebraic equation. Indeed, let μ_j satisfy (51) and set

$$V(t) \stackrel{\text{def}}{=} \begin{cases} 1 & t \leq j_r \\ \mu_0^2(t) & j_r \leq t \leq 0 \\ \mu_j^2 & 0 \leq t, \end{cases}$$

where $\mu_0(t)$ is determined as the solution of

$$\mu_0^4(t)(1 - \mu_0^2(t)) = (t - j_r)^2,$$

satisfying (50). We then look for a solution in $H^1(\mathbb{R}_+)$ of the problem

$$\begin{cases} -w'' + V(w')w = 0 & \text{in } \mathbb{R}_+, \\ w'(0) = j_r. \end{cases} \quad (52)$$

Note that the equation in (52) can be written as $\frac{w''}{V(w')} = w$ and it has a corresponding variational formulation. Indeed, set

$$L(p) = \int_0^p \frac{p-t}{V(t)} dt,$$

so that $L''(p) = \frac{1}{V(p)} \geq 1$ and consider the functional

$$\mathcal{J}(w) = kw(0) + \int_0^\infty \left(L(w') + \frac{w^2}{2} \right), \quad w \in H^1(\mathbb{R}_+), \quad (53)$$

where $k = -L'(j_r)$. Any critical point of \mathcal{J} is a solution of (52) and since \mathcal{J} is strictly convex, there is at most one such critical point—the unique global minimizer for \mathcal{J} . Thanks to the convexity of \mathcal{J} , in order to prove existence of a minimizer it is enough to verify that \mathcal{J} is coercive. But this is evident from the inequality

$$\mathcal{J}(w) \geq \frac{1}{2}\|w\|_{1,2}^2 - |k|c_0\|w\|_{1,2},$$

that can be obtained by observing that $L(p) \geq \frac{p^2}{2}$ and appealing to the Sobolev inequality in one dimension

$$\|v\|_\infty \leq c_0 \|v\|_{1,2}, \quad v \in H^1(\mathbb{R}_+). \quad (54)$$

It remains to show that the solution w to (52) also solves (49). In fact, it would be sufficient to show that

$$w'(\eta) \in [j_r, 0], \quad \eta \in [0, \infty). \quad (55)$$

First we establish that $w \geq 0$ in \mathbb{R}_+ . To this end, because $w \in H^1(\mathbb{R}_+)$ it vanishes at infinity

$$\lim_{\eta \rightarrow \infty} w(\eta) = 0. \quad (56)$$

The function w cannot have a negative minimum in \mathbb{R}_+ because w satisfies (52) and $V > 0$. It then follows from (51) that $w'' \geq 0$ in \mathbb{R}_+ and w' is increasing from j_r to 0.

For future use we note that, taking into account (50), we have that

$$-w'' + \frac{2}{3}w < 0. \quad (57)$$

Then the bound $w(\eta) < w(0)e^{-\sqrt{\frac{2}{3}}\eta}$ for $\eta > 0$ can be deduced via the maximum principle. Further, multiplying the inequality (57) by w' , integrating over \mathbb{R}_+ and using (56), we conclude that $w(0) < -\sqrt{3/2}j_r$ and

$$w(\eta) < \vartheta_m(\eta) := -\sqrt{\frac{3}{2}}j_re^{-\sqrt{\frac{2}{3}}\eta}, \quad (58)$$

for all $\eta > 0$.

Finally, note that we must have $\mu_0(0) = \mu_j$ and that by taking the derivative of the first equation in (49) it follows that $\mu'_0(0) = 0$.

Step 2: For a sufficiently large σ_0 prove the existence of a solution $(\rho_{i0}, \varphi_{i0}) \in L^2(\mathbb{R}_+, \mathbb{R}^2)$ of (48).

Set

$$\mu_1 = \mu - \mu_0; \quad \vartheta_1 = \vartheta - \vartheta_0.$$

We then use (49) to rewrite (48) in the form

$$\begin{cases} -\frac{1}{\sigma_0}\mu_1'' + (6\mu_0^2 - 4)\mu_1 + 2\frac{[1-\mu_0^2]^{1/2}}{\mu_0}\vartheta_1' = \frac{1}{\sigma_0}\mu_0'' + N_1(\mu_1, \vartheta_1) & \text{in } \mathbb{R}_+, \\ -\vartheta_1'' + \mu_0^2\vartheta_1 + 2\mu_0\mu_1\vartheta_0 = N_2(\mu_1, \vartheta_1) & \text{in } \mathbb{R}_+, \\ \mu_1'(0) = 0, \\ \vartheta_1'(0) = 0. \end{cases}$$

Here

$$\begin{aligned} N_1(\mu_1, \vartheta_1) = & -|\vartheta_1'|^2\mu_0^{-3} + (\mu^{-3} - \mu_0^{-3}) \left(\mu^4 - (\vartheta' - j_r)^2 - \mu^6 \right) \\ & + \mu_1\mu_0^{-3} (\mu\mu_0^2 + \mu^2\mu_0 + \mu^3 - 3\mu_0^3 - \mu^5 - \mu^4\mu_0 - \mu^3\mu_0^2 - \mu^2\mu_0^3 - \mu\mu_0^4 + 5\mu_0^5) \end{aligned} \quad (59a)$$

and

$$N_2(\mu_1, \vartheta_1) = -\mu_1^2 \vartheta_0 - (2\mu_0 + \mu_1)\mu_1 \vartheta_1. \quad (59b)$$

Suppose that $\mathbf{U} = (U_1, U_2) \in H^2(\mathbb{R}_+, \mathbb{R}^2)$ and define the operators

$$\begin{cases} \mathcal{L}_1(\mathbf{U}) = -\frac{1}{\sigma_0} U_1'' + (6\mu_0^2 - 4)U_1 + 2\frac{[1 - \mu_0^2]^{1/2}}{\mu_0} U_2' & \text{in } \mathbb{R}_+, \\ \mathcal{L}_2(\mathbf{U}) = -U_2'' + \mu_0^2 U_2 + 2\mu_0 \vartheta_0 U_1 & \text{in } \mathbb{R}_+. \end{cases} \quad (60a)$$

$$\quad \quad \quad (60b)$$

Set $\mathcal{B} : D(\mathcal{B}) \rightarrow L^2(\mathbb{R}_+, \mathbb{R}^2)$ to be defined by

$$\mathcal{B} \mathbf{U} = \begin{bmatrix} \mathcal{L}_1(\mathbf{U}) \\ \mathcal{L}_2(\mathbf{U}) \end{bmatrix}, \quad (61)$$

where

$$D(\mathcal{B}) = \{\mathbf{U} \in H^2(\mathbb{R}_+, \mathbb{R}^2) \mid \mathbf{U}'(0) = 0\}.$$

It can be easily verified, that there exists $\lambda \leq 0$, for which the bilinear form

$$B(\mathbf{U}, \mathbf{V}) = \langle (\mathcal{B} - \lambda \text{Id}) \mathbf{U}, \mathbf{V} \rangle = \int_{\mathbb{R}_+} (\mathcal{B} \mathbf{U} - \lambda \mathbf{U}) \cdot \mathbf{V},$$

is coercive in $H^1(\mathbb{R}_+, \mathbb{R}^2)$. As

$$|B(\mathbf{U}, \mathbf{V})| \leq \|\mathbf{U}\|_2 \|\mathbf{V}\|_2,$$

it follows by the Lax-Milgram lemma that $(\mathcal{B} - \lambda \text{Id})^{-1}$ is bounded.

Let then $\mathbf{U}_\lambda = (U_{\lambda 1}, U_{\lambda 2}) = (\mathcal{B} - \lambda \text{Id})^{-1} \mathbf{F}$ where $\mathbf{F}^t = [F_1, F_2]$ and F_1 and F_2 are both in $L^2(\mathbb{R}_+)$. By (60a) we have that

$$U_{\lambda 1} = -\frac{2[1 - \mu_0^2]^{1/2}}{\mu_0(6\mu_0^2 - 4 - \lambda)} U_{\lambda 2}' + \frac{1}{\sigma_0(6\mu_0^2 - 4 - \lambda)} U_{\lambda 1}'' + \frac{1}{(6\mu_0^2 - 4 - \lambda)} F_1.$$

Substituting this expression into (60b) yields

$$-U_{\lambda 2}'' - \frac{4\vartheta_0[1 - \mu_0^2]^{1/2}}{6\mu_0^2 - 4 - \lambda} U_{\lambda 2}' + (\mu_0^2 - \lambda) U_{\lambda 2} = -\frac{2\mu_0 \vartheta_0}{\sigma_0(6\mu_0^2 - 4 - \lambda)} (U_{\lambda 1}'' + \sigma_0 F_1) + F_2 \quad (62)$$

Let

$$G(\eta, \lambda) = \exp \left\{ \int_0^\eta \frac{4\vartheta_0[1 - \mu_0^2]^{1/2}}{6\mu_0^2 - 4 - \lambda} d\xi \right\}.$$

By (50), the fact that $\lambda \leq 0$, and since the function $\vartheta_0 \geq 0$ in \mathbb{R}_+ , we have that $G \geq 1$. To establish an upper bound for G we first set

$$\delta = \inf_{\eta \in \mathbb{R}_+} 6\mu_0^2(\eta) - 4,$$

and use (49) and (50) to note that $\delta > 0$ if $j_r^2 < 4/27$. Since $\vartheta_0 \leq \vartheta_m$ on \mathbb{R}_+ —where ϑ_m is defined in (58)—and using (50) we obtain that

$$G \leq \exp \left\{ \frac{4}{\sqrt{3}(\delta - \lambda)} \int_0^\infty \vartheta_m(\eta) d\eta \right\} = \exp \left\{ -\frac{2\sqrt{3}j_r}{\delta - \lambda} \right\},$$

i.e., there exists a constant $C > 0$, independent of δ and λ , such that

$$G \leq \exp \left\{ \frac{C}{\delta - \lambda} \right\}. \quad (63)$$

To simplify notation, in the remainder of this argument C will denote a generic constant independent of δ and λ .

We now multiply (62) by $GU_{\lambda 2}$ and integrate by parts to obtain that

$$\begin{aligned} \|G^{1/2}U'_{\lambda 2}\|_2^2 + \|G^{1/2}\sqrt{\mu_0^2 - \lambda}U_{\lambda 2}\|_2^2 &= \left\langle \frac{2\mu_0\vartheta_0}{\sigma_0(6\mu_0^2 - 4 - \lambda)}GU'_{\lambda 2}, U'_{\lambda 1} \right\rangle + \\ &\left\langle \left(\frac{2\mu_0\vartheta_0}{\sigma_0(6\mu_0^2 - 4 - \lambda)}G \right)' U_{\lambda 2}, U'_{\lambda 1} \right\rangle - \left\langle \frac{2\mu_0\vartheta_0}{6\mu_0^2 - 4 - \lambda}F_1, GU_{\lambda 2} \right\rangle + \langle F_2, GU_{\lambda 2} \rangle. \end{aligned}$$

This inequality with the aid of (63) allows us to conclude that

$$\|U'_{\lambda 2}\|_2 + \|U_{\lambda 2}\|_2 \leq e^{\frac{C}{\delta - \lambda}} \left(\|\mathbf{F}\|_2 + \frac{1}{\sigma_0} \|U'_{\lambda 1}\|_2 \right). \quad (64)$$

We next observe that

$$\begin{aligned} \langle U_{\lambda 1}, \mathcal{L}_1(\mathbf{U}_\lambda) - \lambda U_{\lambda 1} \rangle &= \frac{1}{\sigma_0} \|U'_{\lambda 1}\|_2^2 + \|(6\mu_0^2 - 4)^{1/2}U_{\lambda 1}\|_2^2 \\ &\quad - \lambda \|U_{\lambda 1}\|_2^2 + \left\langle 2 \frac{[1 - \mu_0^2]^{1/2}}{\mu_0} U'_{\lambda 2}, U_{\lambda 1} \right\rangle = \langle F_1, U_{\lambda 1} \rangle. \end{aligned}$$

With the aid of (64) we then obtain,

$$\frac{1}{\sigma_0} \|U'_{\lambda 1}\|_2^2 + \delta \|U_{\lambda 1}\|_2^2 \leq e^{\frac{C}{\delta - \lambda}} \left(\|\mathbf{F}\|_2 \|U_{\lambda 1}\|_2 + \frac{1}{\sigma_0} \|U'_{\lambda 1}\|_2 \|U_{\lambda 1}\|_2 \right),$$

from which we easily conclude that for some $C > 0$,

$$\|U_{\lambda 1}\|_2 \leq e^{\frac{C}{\delta - \lambda}} \left(\frac{1}{\sigma_0} \|U'_{\lambda 1}\|_2 + \|\mathbf{F}\|_2 \right).$$

Consequently, we obtain that

$$\frac{1}{\sigma_0^{1/2}} \|U'_{\lambda 1}\|_2 + \|U_{\lambda 1}\|_2 \leq e^{\frac{C}{\delta - \lambda}} \|\mathbf{F}\|_2.$$

Combining the above with (64) then yields

$$\frac{1}{\sigma_0^{1/2}} \|U'_{\lambda 1}\|_2 + \|U'_{\lambda 2}\|_2 + \|\mathbf{U}\|_2 \leq e^{\frac{C}{\delta - \lambda}} \|\mathbf{F}\|_2. \quad (65)$$

It is well known that the real resolvent set $\rho(\mathcal{B}) \cap \mathbb{R}$ is open. Furthermore, by (65) $\rho(\mathcal{B}) \cap (-\infty, \delta/2)$ must be closed. Hence $(-\infty, \delta/2) \subset \rho(\mathcal{B})$ and, in particular, $0 \in \rho(\mathcal{B})$. It follows that

$$\mathbf{U} = \mathcal{B}^{-1}\mathbf{F} \quad (66)$$

is well defined and satisfies

$$\frac{1}{\sigma_0^{1/2}} \|U'_1\|_2 + \|U_1\|_2 + \|U_2\|_{2,2} \leq C_\delta \|\mathbf{F}\|_2, \quad (67)$$

by (60) and (65).

Let now $(\mu_1, \vartheta_1) \in \mathcal{W}$ where $\mathcal{W} = H^1(\mathbb{R}_+) \times H^2(\mathbb{R}_+)$ is equipped with the norm

$$\|(u_1, u_2)\|_{\mathcal{W}} = \|u_1\|_2 + \frac{1}{\sigma_0^{1/2}} \|u'_1\|_2 + \|u_2\|_{2,2}, \quad (68)$$

and note that (67) can be rewritten as

$$\|\mathcal{B}^{-1}\mathbf{F}\|_{\mathcal{W}} \leq C_\delta \|\mathbf{F}\|_2. \quad (69)$$

Next, let r satisfy

$$r = \sigma_0^{-\alpha} \text{ for some } 3/4 < \alpha < 1. \quad (70)$$

Consider $(\mu_1, \vartheta_1) \in B(0, r)$ and set

$$F_1 = \frac{1}{\sigma_0} \mu''_0 + N_1(\mu_1, \vartheta_1); \quad F_2 = N_2(\mu_1, \vartheta_1), \quad (71)$$

with N_1 and N_2 as defined in (59). Applying the Sobolev inequality (54) with $v = \mu_1$, using (70), yields

$$\|\mu_1\|_\infty \leq C \|\mu_1\|_{1,2} \leq C \sigma_0^{1/2} r \leq \sigma_0^{-1/4}, \quad (72)$$

for sufficiently large σ_0 . In particular,

$$\mu \in (\mu_0/2, 3\mu_0/2). \quad (73)$$

Similarly,

$$\begin{aligned} \|\vartheta'_1\|_\infty &\leq C \|\vartheta'_1\|_{1,2} \leq Cr \leq \sigma_0^{-3/4}, \\ \|\vartheta_1\|_\infty &\leq C \|\vartheta_1\|_{1,2} \leq Cr \leq \sigma_0^{-3/4}. \end{aligned} \quad (74)$$

Using (73)-(74) in (59) yields,

$$\begin{aligned} |N_1(\mu_1, \vartheta_1)| &\leq c_1 |\mu_1|^2 + c_2 |\vartheta'_1|^2, \\ |N_2(\mu_1, \vartheta_1)| &\leq c_3 |\mu_1|^2 + c_4 |\vartheta_1|^2, \end{aligned} \quad (75)$$

for some constants c_1, \dots, c_4 . Now, with the aid of (72) and (74) equation (75) gives

$$\begin{aligned} \|N_1(\mu_1, \vartheta_1)\|_2 &\leq C (\|\mu_1\|_2 \|\mu_1\|_\infty + \|\vartheta'_1\|_2 \|\vartheta'_1\|_\infty) \\ &\leq C \|\mu_1\|_2 (\|\mu_1\|_2 + \|\mu'_1\|_2) + C \|\vartheta'_1\|_2 (\|\vartheta'_1\|_2 + \|\vartheta''_1\|_2) \\ &\leq C \|(\mu_1, \vartheta_1)\|_{\mathcal{W}}^2 (\sigma_0^{1/2} + 1). \end{aligned} \quad (76)$$

By a similar argument also

$$\|N_2(\mu_1, \vartheta_1)\|_2 \leq C\|(\mu_1, \vartheta_1)\|_{\mathcal{W}}^2(\sigma_0^{1/2} + 1). \quad (77)$$

By (76)–(77) and (71) we obtain, for a sufficiently large σ_0 , that

$$\|\mathbf{F}\|_2 \leq C\left(\sigma_0^{1/2}r^2 + \frac{1}{\sigma_0}\right). \quad (78)$$

We now define the non-linear operator $\mathcal{A} : \mathcal{W} \rightarrow \mathcal{W}$ by $\mathcal{A}(\mu_1, \vartheta_1) = \mathbf{U}$, where \mathbf{U} is defined via (66) and \mathbf{F} given by (71).

We first show that $\mathcal{A} : B(0, r) \rightarrow B(0, r)$. To this end we use (69), (70), and (78) to obtain that

$$\|\mathbf{U}\|_{\mathcal{W}} \leq C\left(\sigma_0^{1/2}r^2 + \frac{1}{\sigma_0}\right) \leq C\sigma_0^{-1} < r, \quad (79)$$

for a sufficiently large σ_0 , i.e., $\mathcal{A}(\mu_1, \vartheta_1) \in B(0, r)$.

Finally, we prove that \mathcal{A} is a contraction. Let (v_1, w_1) and (v_2, w_2) be in $B(0, r)$. Let $\mathbf{V} = (v_1 - v_2, w_1 - w_2)$. A direct computation, using (59), gives

$$\begin{aligned} & |N_1(v_1, w_1) - N_1(v_2, w_2)| + |N_2(v_1, w_1) - N_2(v_2, w_2)| \leq \\ & C \max(|v_1|, |w'_1|, |w_1|, |v_2|, |w'_2|, |w_2|)(|v_1 - v_2| + |w_1 - w_2| + |w'_1 - w'_2|). \end{aligned} \quad (80)$$

From (72), (74), and (80) we obtain,

$$\|N_1(v_1, w_1) - N_1(v_2, w_2)\|_2 + \|N_2(v_1, w_1) - N_2(v_2, w_2)\|_2 \leq C\sigma_0^{-1/4}\|\mathbf{V}\|_{\mathcal{W}}. \quad (81)$$

Finally, applying (69) and (81) we get that

$$\begin{aligned} & \|\mathcal{A}(v_1, w_1) - \mathcal{A}(v_2, w_2)\|_{\mathcal{W}} = \\ & \left\| \mathcal{B}^{-1}\left(\frac{\mu_0''}{\sigma_0} + N_1(v_1, w_1), N_2(v_1, w_1)\right) - \mathcal{B}^{-1}\left(\frac{\mu_0''}{\sigma_0} + N_1(v_2, w_2), N_2(v_2, w_2)\right) \right\|_{\mathcal{W}} \\ & \leq C\sigma_0^{-1/4}\|\mathbf{V}\|_{\mathcal{W}} \leq \frac{1}{2}\|\mathbf{V}\|_{\mathcal{W}}, \end{aligned}$$

for large enough σ_0 , i.e., $\mathcal{A} : B(0, r) \rightarrow B(0, r)$ is a strict contraction. Applying Banach Fixed Point Theorem completes the proof of existence.

Note that by (79) we have that

$$\|\mu - \mu_0\|_2 + \sigma_0^{-1/2}\|\mu' - \mu'_0\|_2 + \|\vartheta - \vartheta_0\|_{2,2} \leq \frac{C}{\sigma_0}, \quad (82)$$

and consequently

$$\sigma_0^{-1/2}\|\mu - \mu_0\|_{\infty} + \|\vartheta' - \vartheta'_0\|_{\infty} + \|\vartheta - \vartheta_0\|_{\infty} \leq \frac{C}{\sigma_0}. \quad (83)$$

Step 3: Prove (46) and that

$$\mu \geq \mu_j. \quad (84)$$

Since ϑ can have neither a positive maximum nor a negative minimum we obtain that ϑ positive and decreasing, and hence also that $j_r < \vartheta' < 0$. Consequently, if at some point $\sqrt{2/3} < \mu < \mu_j$, we have that

$$1 - \frac{(\vartheta' - j_r)^2}{\mu^4} - \mu^2 > 1 - \frac{j_r^2}{\mu^4} - \mu^2 > 0.$$

Further, $\mu'' < 0$ so that μ cannot have a minimum value between $\sqrt{2/3}$ and μ_j . In view of (82) and since $\mu_0 > \mu_j$ in \mathbb{R}_+ by (49), we obtain that for a sufficiently large σ_0 , the equation (84) must be satisfied.

A standard comparison argument along with the Hopf lemma now shows that

$$\vartheta \leq -\frac{j_r}{\mu_j} e^{-\mu_j \eta},$$

in \mathbb{R}_+ . Further, since $\mu < 1$, integrating the second equation in (48) gives

$$\vartheta'(\eta) = - \int_{\eta}^{\infty} \mu^2 \vartheta d\tilde{\eta} \geq \int_{\eta}^{\infty} \frac{j_r}{\mu_j} e^{-\mu_j \tilde{\eta}} d\tilde{\eta} = \frac{j_r}{\mu_j^2} e^{-\mu_j \eta}.$$

It follows that

$$|\vartheta| + \mu_j |\vartheta'| \leq -\frac{2j_r}{\mu_j} e^{-\mu_j \eta},$$

To prove exponential decay of $\mu - \mu_j$ we first observe that

$$-\left(1 - \frac{(\vartheta' - j_r)^2}{\mu^4} - \mu^2\right)\mu > (6\mu_j^2 - 4)(\mu - \mu_j) + Ce^{-\mu_j \eta}.$$

Standard comparison arguments and (41) complete the proof of (46) with the help of (47). \square

For later reference, we need to obtain some estimates on the derivatives of μ and ϑ with respect to s , which is merely a parameter in (48). Let then $\tilde{\mu} = \mu - \mu_j$. Taking the derivative of (48) with respect to s yields

$$\begin{cases} -\frac{1}{\sigma_0} \left(\frac{\partial \tilde{\mu}}{\partial s} \right)'' + \left(6\mu^2 - 4 - \frac{3}{\sigma_0} \frac{\mu''}{\mu} \right) \frac{\partial \tilde{\mu}}{\partial s} + 2 \frac{\vartheta' - j_r}{\mu^3} \frac{\partial \vartheta'}{\partial s} \\ \quad = - \left(6\mu^2 - 4 - \frac{3}{\sigma_0} \frac{\mu''}{\mu} \right) \frac{\partial \mu_j}{\partial s} + 2 \frac{\vartheta' - j_r}{\mu^3} \frac{\partial j_r}{\partial s}, \\ - \left(\frac{\partial \vartheta}{\partial s} \right)'' + \mu^2 \left(\frac{\partial \vartheta}{\partial s} \right)' + 2\mu \vartheta \frac{\partial \tilde{\mu}}{\partial s} = -2\mu \vartheta \frac{\partial \mu_j}{\partial s}, \\ \left(\frac{\partial \tilde{\mu}}{\partial s} \right)'(0) = 0, \\ \left(\frac{\partial \vartheta}{\partial s} \right)'(0) = \frac{\partial j_r}{\partial s}, \end{cases} \quad (85)$$

after some manipulations where (48) is used once again.

We can now prove the following

Lemma 3.2. Let $\left(\frac{\partial \tilde{\mu}}{\partial s}, \frac{\partial \vartheta}{\partial s}\right)$ denote a solution of (85). Then, there exists some $C(j_r) > 0$ such that,

$$\left\| \frac{\partial \tilde{\mu}}{\partial s} \right\|_{C^2(\mathbb{R}_+)} + \left\| \frac{\partial \vartheta}{\partial s} \right\|_{C^2(\mathbb{R}_+)} \leq C. \quad (86)$$

Furthermore, there exists some $\gamma > 0$ such that

$$\left| \frac{\partial \vartheta'}{\partial s} \right| + \left| \frac{\partial \tilde{\mu}}{\partial s} \right| \leq C e^{-\gamma \eta}. \quad (87)$$

Proof. In order to replace the system in (85) by a system with homogeneous boundary conditions we change variables and let

$$\frac{\partial \tilde{\vartheta}}{\partial s} = \frac{\partial \vartheta}{\partial s} - \frac{\partial j_r}{\partial s} e^{-\mu_j \eta}.$$

We now represent (85) in the following manner

$$\begin{cases} \mathcal{L}_1 \left(\frac{\partial \tilde{\mu}}{\partial s}, \frac{\partial \tilde{\vartheta}}{\partial s} \right) = f_1 \text{ in } \mathbb{R}_+, \end{cases} \quad (88a)$$

$$\begin{cases} \mathcal{L}_2 \left(\frac{\partial \tilde{\mu}}{\partial s}, \frac{\partial \tilde{\vartheta}}{\partial s} \right) = f_2 \text{ in } \mathbb{R}_+, \end{cases} \quad (88b)$$

$$\begin{cases} \left(\frac{\partial \tilde{\mu}}{\partial s} \right)'(0) = 0, \end{cases} \quad (88c)$$

$$\begin{cases} \left(\frac{\partial \tilde{\vartheta}}{\partial s} \right)'(0) = 0, \end{cases} \quad (88d)$$

where

$$\begin{aligned} f_1 &= \left(6[\mu_j^2 - \mu^2] + \frac{3}{\sigma_0} \frac{\mu''}{\mu} \right) \frac{\partial \mu_j}{\partial s} + 2 \left[\frac{j_r}{\mu_j^3} - \frac{j_r}{\mu^3} + \frac{\vartheta'}{\mu^3} \right] \frac{\partial j_r}{\partial s} + \\ &\quad \left[6(\mu_0^2 - \mu^2) + \frac{3}{\sigma_0} \frac{\mu''}{\mu} \right] \frac{\partial \tilde{\mu}}{\partial s} + 2 \left[\frac{[1 - \mu_0^2]^{1/2}}{\mu_0} - \frac{\vartheta' - j_r}{\mu^3} \right] \frac{\partial \tilde{\vartheta}'}{\partial s} + 2\mu_j \frac{\vartheta' - j_r}{\mu^3} \frac{\partial j_r}{\partial s} e^{-\mu_j \eta}, \\ f_2 &= -2\mu \vartheta \frac{\partial \mu_j}{\partial s} + (\mu_0^2 - \mu^2) \frac{\partial \tilde{\vartheta}}{\partial s} + (\mu_j^2 - \mu^2) \frac{\partial j_r}{\partial s} e^{-\mu_j \eta} + 2(\mu_0 \vartheta_0 - \mu \vartheta) \frac{\partial \tilde{\mu}}{\partial s}, \end{aligned}$$

and the operators \mathcal{L}_1 and \mathcal{L}_2 are as defined in (60). Here, in order to obtain the expression for f_1 , we have also used the derivative with respect to s of the equation (51).

Equivalently, we can represent (88) in the form

$$\mathcal{B}V = \tilde{\mathcal{B}}V + F, \quad (89)$$

where \mathcal{B} is given by (61) while

$$V = \begin{bmatrix} \partial \tilde{\mu} / \partial s \\ \partial \tilde{\vartheta} / \partial s \end{bmatrix}, \quad F = \begin{bmatrix} \left(6[\mu_j^2 - \mu^2] + \frac{3}{\sigma_0} \frac{\mu''}{\mu} \right) \frac{\partial \mu_j}{\partial s} + 2 \left[\frac{j_r}{\mu_j^3} - \frac{j_r}{\mu^3} + \frac{\vartheta'}{\mu^3} \right] \frac{\partial j_r}{\partial s} + 2\mu_j \frac{\vartheta' - j_r}{\mu^3} \frac{\partial j_r}{\partial s} e^{-\mu_j \eta} \\ -2\mu \vartheta \frac{\partial \mu_j}{\partial s} + (\mu_j^2 - \mu^2) \frac{\partial j_r}{\partial s} e^{-\mu_j \eta} \end{bmatrix},$$

and

$$\tilde{\mathcal{B}} = \begin{bmatrix} 6(\mu_0^2 - \mu^2) + \frac{3}{\sigma_0} \frac{\mu''}{\mu} & 2 \left[\frac{[1 - \mu_0^2]^{1/2}}{\mu_0} - \frac{\vartheta' - j_r}{\mu^3} \right] \frac{d}{d\eta} \\ 2(\mu_0 \vartheta_0 - \mu \vartheta) & (\mu_0^2 - \mu^2) \end{bmatrix}.$$

By (46) and (48) we have that for some $\gamma_0 > 0$

$$\|e^{\gamma_0 \eta} F\|_2 < \infty. \quad (90)$$

In view of (48), (49) and (83) we have that

$$\left| \frac{1}{\sigma_0} \frac{\mu''}{\mu} \right| = \left| 1 - \frac{(\vartheta' - j_r)^2}{\mu^4} - \mu^2 \right| \leq \frac{C}{\sigma_0^{1/2}}. \quad (91)$$

Furthermore, by (49) and (83) we have that

$$\left| \frac{[1 - \mu_0^2]^{1/2}}{\mu_0} - \frac{\vartheta' - j_r}{\mu^3} \right| \leq \frac{C}{\sigma_0^{1/2}}. \quad (92)$$

Hence, with the aid of (83) we obtain that

$$\|\tilde{\mathcal{B}}V\|_2 \leq \frac{C}{\sigma_0^{1/2}} \left[\|V\|_2 + \left\| \frac{\partial \tilde{\vartheta}'}{\partial s} \right\|_2 \right] \leq \frac{C}{\sigma_0^{1/2}} \|V\|_{\mathcal{W}}.$$

By (69) we now obtain,

$$\|V\|_{\mathcal{W}} \leq C(\|\tilde{\mathcal{B}}V\|_2 + \|F\|_2) \leq \frac{C}{\sigma_0^{1/2}} \|V\|_{\mathcal{W}} + C\|F\|_2,$$

whence

$$\|V\|_{\mathcal{W}} \leq C\|F\|_2. \quad (93)$$

Using the above and a standard ODE regularity argument we can easily prove (86).

To prove (87) we take the inner product of (88b) in $L^2(\mathbb{R}_+)$ with $e^{2\gamma\eta} \frac{\partial \tilde{\vartheta}}{\partial s}$ to obtain, using the fact that $\mu \geq \mu_j$ by (84),

$$\begin{aligned} & \left\| \left(e^{\gamma\eta} \frac{\partial \tilde{\vartheta}}{\partial s} \right)' \right\|_2^2 + (\mu_j^2 - \gamma^2) \left\| e^{\gamma\eta} \frac{\partial \tilde{\vartheta}}{\partial s} \right\|_2^2 \\ & \leq \left\| e^{\gamma\eta} \frac{\partial \tilde{\vartheta}}{\partial s} \right\|_2 \left(\left\| e^{\gamma\eta} 2\mu\vartheta \left[\frac{\partial \tilde{\mu}}{\partial s} + \frac{\partial \mu_j}{\partial s} \right] \right\|_2 + \left\| e^{\gamma\eta} (\mu_j^2 - \mu^2) \frac{\partial j_r}{\partial s} e^{-\mu_j \eta} \right\|_2 \right). \end{aligned}$$

Hence, by (46) and (93),

$$\left\| \left(e^{\gamma\eta} \frac{\partial \tilde{\vartheta}}{\partial s} \right)' \right\|_2^2 + \left\| e^{\gamma\eta} \frac{\partial \tilde{\vartheta}}{\partial s} \right\|_2^2 \leq C, \quad (94)$$

for a sufficiently small γ . Taking the inner product of (88a) in $L^2(\mathbb{R}_+)$ with $e^{2\gamma\eta}\frac{\partial\tilde{\mu}}{\partial s}$ for some $\gamma < \gamma_0$ yields

$$\begin{aligned} \frac{1}{\sigma_0^2} \left\| \left(e^{\gamma\eta} \frac{\partial\tilde{\mu}}{\partial s} \right)' \right\|_2^2 + (6\mu_j^2 - 4 - \gamma^2) \left\| e^{\gamma\eta} \frac{\partial\tilde{\mu}}{\partial s} \right\|_2^2 \\ \leq \left\| e^{\gamma\eta} \frac{\partial\tilde{\mu}}{\partial s} \right\|_2 \left(C \left\| e^{\gamma\eta} \left(\frac{\partial\tilde{\vartheta}}{\partial s} \right)' \right\|_2 + \|e^{\gamma\eta} f_1\|_2 \right). \end{aligned}$$

With the aid of (90), (94), (91), and (92), we than obtain that

$$\frac{1}{\sigma_0^2} \left\| \left(e^{\gamma\eta} \frac{\partial\tilde{\mu}}{\partial s} \right)' \right\|_2^2 + \left\| e^{\gamma\eta} \frac{\partial\tilde{\mu}}{\partial s} \right\|_2^2 \leq C.$$

Sobolev embeddings then provide (87). Note that the constant C in (87) may depend on σ_0 , a fact that shouldn't be of any concern to us, since in the sequel we keep σ_0 fixed (though sufficiently large) while letting $\epsilon \rightarrow 0$. \square

Remark 3.1. Repeating the procedure outlined above, one can similarly obtain

$$\left\| \frac{\partial^2 \mu}{\partial s^2} \right\|_{C^2(\mathbb{R}_+)} + \left\| \frac{\partial^2 \vartheta}{\partial s^2} \right\|_{C^2(\mathbb{R}_+)} \leq C, \quad (95)$$

and that

$$\left| \frac{\partial^2 \tilde{\mu}}{\partial s^2} \right| + \left| \frac{\partial^2 \vartheta'}{\partial s^2} \right| \leq C e^{-\gamma\eta}. \quad (96)$$

Finally, by (41) we obtain the bound

$$\left\| \frac{\partial v'_{i0}}{\partial s} \right\|_{C^1(\mathbb{R}_+)} + \left\| \frac{\partial^2 v'_{i0}}{\partial s^2} \right\|_{C^1(\mathbb{R}_+)} \leq C$$

so that by (40c), (46), (86), and (87) we have that

$$\left| \frac{\partial v'_{i0}}{\partial s} \right| \leq C e^{-\gamma\eta}. \quad (97)$$

To prove that the above solution of (42) is indeed a good inner approximation, we need to determine the next order term in the inner expansion. Let then $(\rho_{i1}, \varphi_{i1}, v_{i1})$ denote the solution of

$$\begin{cases} -\rho''_{i1} - \left(\rho_r^2 - \left| v'_{i0} + \frac{\partial\zeta}{\partial t}(s, 0) \right|^2 - 3\rho_{i0}^2 \right) \rho_{i1} + 2\rho_{i0} \left(v'_{i0} + \frac{\partial\zeta}{\partial t}(s, 0) \right) v'_{i1} \\ \quad = R_\rho(s, \tau) & \text{in } \mathbb{R}_+, \quad (98a) \\ -\sigma_0 \varphi''_{i1} + \rho_{i0}^2 \varphi_{i1} + 2\rho_{i0} \rho_{i1} \varphi_{i0} = \sigma_0 \kappa \varphi'_{i0} & \text{in } \mathbb{R}_+, \quad (98b) \\ (\rho_{i0}^2 v'_{i1})' + 2 \left(\rho_{i0} \rho_{i1} \left(v'_{i0} + \frac{\partial\zeta}{\partial t}(s, 0) \right) \right)' - \sigma_0 \varphi''_{i1} = R_v(s, \tau) & \text{in } \mathbb{R}_+, \quad (98c) \\ \rho'_{i1}(0) = 0, & (98d) \\ \varphi'_{i1}(0) = 0, & (98e) \\ v'_{i1}(0) = 0, & (98f) \end{cases}$$

where

$$R_\rho(s, \tau) := -\kappa \rho'_{i0} - 2\rho_{i0}\tau \left(v'_{i0} + \frac{\partial \zeta}{\partial t}(s, 0) \right) \frac{\partial^2 \zeta}{\partial t^2}(s, 0) \\ - 2\rho_{i0} \left(\kappa \tau \frac{\partial \zeta}{\partial s}(s, 0) + \tau \frac{\partial^2 \zeta}{\partial s \partial t}(s, 0) + \frac{\partial v_{i0}}{\partial s} \right) \frac{\partial \zeta}{\partial s}(s, 0), \quad (99)$$

and

$$R_v(s, \tau) := -\kappa j - \frac{\partial}{\partial s} \left(\rho_{i0}^2 \frac{\partial \zeta}{\partial s}(s, 0) \right) - \left(\rho_{i0}^2 \tau \frac{\partial^2 \zeta}{\partial t^2}(s, 0) \right)'.$$

This system is obtained by first extracting terms of order ϵ in (38) and using (41).

In what follows, we seek estimates of the right hand sides of the equations (98a)-(98c). We will need the following set of identities.

Lemma 3.3. *Suppose that ζ , j and ρ_j are as defined in (15) and (45), respectively. Then*

$$\kappa \left| \frac{\partial \zeta}{\partial s} \right|^2 + \frac{\partial \zeta}{\partial t} \frac{\partial^2 \zeta}{\partial t^2} + \frac{\partial \zeta}{\partial s} \frac{\partial^2 \zeta}{\partial t \partial s} - \frac{1}{2} \frac{\partial}{\partial t} |\nabla \zeta|^2 = 0 \quad (100)$$

and

$$\kappa j + \frac{\partial}{\partial s} \left(\rho_j^2 \frac{\partial \zeta}{\partial s} \right) + \frac{\partial}{\partial t} \left(\rho_j^2 \frac{\partial \zeta}{\partial t} \right) = 0 \quad (101)$$

hold on $\partial\Omega$.

Proof. First, use (36) to observe that

$$\frac{\partial}{\partial t} |\nabla \zeta|^2 = \frac{\partial}{\partial t} \left(\frac{1}{\mathfrak{g}} \left| \frac{\partial \zeta}{\partial s} \right|^2 + \left| \frac{\partial \zeta}{\partial t} \right|^2 \right) = \frac{\kappa}{\mathfrak{g}^2} \left| \frac{\partial \zeta}{\partial s} \right|^2 + \frac{2}{\mathfrak{g}} \frac{\partial \zeta}{\partial s} \frac{\partial^2 \zeta}{\partial t \partial s} + \frac{2}{\mathfrak{g}} \frac{\partial \zeta}{\partial t} \frac{\partial^2 \zeta}{\partial t^2},$$

then (100) follows by setting $t = 0$. To establish (101) note that

$$\frac{\partial}{\partial s} \left(\rho_j^2 \frac{\partial \zeta}{\partial s}(s, 0) \right) = -\frac{1}{\mathfrak{g}} \frac{\partial}{\partial t} \left(\mathfrak{g}(1 - |\nabla \zeta|^2) \frac{\partial \zeta}{\partial t} \right)_{t=0},$$

by (15). Taking the derivative on the right hand side of this equation gives (101). \square

We now set

$$\tilde{\rho}_{i1} = \rho_{i1} + \frac{1}{2\rho_j} \frac{\partial}{\partial t} |\nabla \zeta|^2|_{(s,0)} \tau$$

and, taking into account (41), obtain that

$$\left\{ \begin{array}{ll} -\tilde{\rho}_{i1}'' - \left(\rho_r^2 - \frac{(\sigma_0 \varphi'_{i0} - j)^2}{\rho_{i0}^4} - 3\rho_{i0}^2 \right) \tilde{\rho}_{i1} + 2\rho_{i0} \left(v'_{i0} + \frac{\partial \zeta}{\partial t}(s, 0) \right) v'_{i1} \\ \quad = \tilde{R}_\rho(s, \tau) & \text{in } \mathbb{R}_+, \quad (102a) \\ -\sigma_0 \varphi_{i1}'' + \rho_{i0}^2 \varphi_{i1} + 2\rho_{i0} \tilde{\rho}_{i1} \varphi_{i0} = \tilde{R}_\varphi & \text{in } \mathbb{R}_+, \quad (102b) \\ \left(\rho_{i0}^2 v'_{i1} \right)' + 2 \left(\rho_{i0} \tilde{\rho}_{i1} \left[v'_{i0} + \frac{\partial \zeta}{\partial t}(s, 0) \right] \right)' - \sigma_0 \varphi_{i1}'' = \tilde{R}_v(s, \tau) & \text{in } \mathbb{R}_+, \quad (102c) \\ \tilde{\rho}'_{i1}(0) = -\frac{1}{2\rho_j} \frac{\partial}{\partial t} |\nabla \zeta|^2|_{(s,0)}, & (102d) \\ \varphi'_{i1}(0) = 0, & (102e) \\ v'_{i1}(0) = 0, & (102f) \end{array} \right.$$

where

$$\begin{aligned} \tilde{R}_\rho(s, \tau) &= -\kappa \rho'_{i0} - 2(\rho_{i0} - \rho_j) \tau \frac{\partial \zeta}{\partial t}(s, 0) \frac{\partial^2 \zeta}{\partial t^2}(s, 0) - 2\rho_{i0} \tau v'_{i0} \frac{\partial^2 \zeta}{\partial t^2}(s, 0) \\ &\quad - 2(\rho_{i0} - \rho_j) \left(\kappa \tau \frac{\partial \zeta}{\partial s}(s, 0) + \tau \frac{\partial^2 \zeta}{\partial s \partial t}(s, 0) \right) \frac{\partial \zeta}{\partial s}(s, 0) - 2\rho_{i0} \frac{\partial v_{i0}}{\partial s} \frac{\partial \zeta}{\partial s}(s, 0) \\ &\quad - \left(\rho_r^2 - \frac{(\sigma_0 \varphi'_{i0} - j)^2}{\rho_{i0}^4} - 3\rho_{i0}^2 + 2\rho_j^2 \right) \frac{1}{2\rho_j} \frac{\partial}{\partial t} |\nabla \zeta|^2|_{(s,0)} \tau, \\ \tilde{R}_\varphi &= \sigma_0 \kappa \varphi'_{i0} + \frac{\rho_{i0}}{\rho_j} \varphi_{i0} \frac{\partial}{\partial t} |\nabla \zeta|^2|_{(s,0)} \tau, \end{aligned}$$

and

$$\begin{aligned} \tilde{R}_v(s, \tau) &= \frac{\partial}{\partial s} \left((\rho_j^2 - \rho_{i0}^2) \frac{\partial \zeta}{\partial s}(s, 0) \right) \\ &\quad + \left(\left[(\rho_j^2 - \rho_{i0}^2) \frac{\partial^2 \zeta}{\partial t^2}(s, 0) + \frac{1}{\rho_j} \left(\rho_{i0} v'_{i0} + (\rho_{i0} - \rho_j) \frac{\partial \zeta}{\partial t}(s, 0) \right) \frac{\partial}{\partial t} |\nabla \zeta|^2|_{(s,0)} \right] \tau \right)'. \end{aligned}$$

Here we have used Lemma 3.3 to express \tilde{R}_ρ and \tilde{R}_φ in terms of $\rho_{i0} - \rho_j$.

We now have the following

Lemma 3.4. *There exists a solution $(\tilde{\rho}_{i1}, \varphi_{i1}, v_{i1})$ for (102) such that*

$$e^{\gamma \tau} [|\tilde{\rho}_{i1}| + |\varphi_{i1}| + |v'_{i1}|] \in L^2(\mathbb{R}_+). \quad (103)$$

Proof. We begin by integrating (102c) to obtain using (46) and (97) that

$$\rho_{i0}^2 v'_{i1} + 2\rho_{i0} \tilde{\rho}_{i1} \left(v'_{i0} + \frac{\partial \zeta}{\partial t}(s, 0) \right) - \sigma_0 \varphi'_{i1} = F(\tau), \quad (104)$$

where

$$F(\tau) = - \int_{\tau}^{\infty} \frac{\partial}{\partial s} \left((\rho_j^2 - \rho_{i0}^2) \frac{\partial \zeta}{\partial s}(s, 0) \right) d\tau' \\ + \left[(\rho_j^2 - \rho_{i0}^2) \frac{\partial^2 \zeta}{\partial t^2}(s, 0) + \frac{1}{\rho_j} \left(\rho_{i0} v'_{i0} + (\rho_{i0} - \rho_j) \frac{\partial \zeta}{\partial t}(s, 0) \right) \frac{\partial}{\partial t} |\nabla \zeta|^2 \Big|_{(s,0)} \right] \tau.$$

Now, using the rescalings (47) and setting

$$\varsigma(\eta) = \frac{1}{\rho_r} \tilde{\rho}_{i1} \left(\frac{\sigma_0^{1/2} \eta}{\rho_r} \right) - \frac{\sigma_0^{1/2}}{\rho_r} \left(\frac{\partial}{\partial t} |\nabla \zeta|^2 \Big|_{(s,0)} \right) \frac{1}{\mu_j} e^{-\mu_j \eta}, \quad \psi(\eta) = \frac{\sigma_0^{1/2}}{\rho_r^2} \varphi_{i1} \left(\frac{\sigma_0^{1/2} \eta}{\rho_r} \right),$$

we substitute v'_{i1} from (104) into (98ab) and (98de) yielding with the help of (41) and (42) the system

$$\begin{cases} -\frac{1}{\sigma_0} \varsigma'' + \left(6\mu^2 - 4 - \frac{3}{\sigma_0} \frac{\mu''}{\mu} \right) \varsigma + 2 \frac{\vartheta' - j_r}{\mu^3} \psi' = \frac{G(\rho_r^{-1} \sigma_0^{1/2} \eta)}{\rho_r^3} & \text{in } \mathbb{R}_+, \\ -\psi'' + \mu^2 \varphi_{i1} + 2\mu \vartheta \varsigma = \frac{\sigma_0^{1/2}}{\rho_r} \left[-2\mu \vartheta \left(\frac{\partial}{\partial t} |\nabla \zeta|^2 \Big|_{(s,0)} \right) \left(\eta + \frac{1}{\mu_j} e^{-\mu_j \eta} \right) + \kappa \vartheta' \right] & \text{in } \mathbb{R}_+, \\ \varsigma'(0) = 0, \\ \psi'(0) = 0. \end{cases} \quad (105)$$

Here

$$G(\tau) = \tilde{R}_\rho - 2 \frac{\sigma_0 \varphi'_{i0} - j_r}{\rho_{i0}^3} F(\tau) + \frac{\rho_r^2}{\sigma_0^{1/2}} \left(\frac{\partial}{\partial t} |\nabla \zeta|^2 \Big|_{(s,0)} \right) \mu_j e^{-\mu_j \rho_r \sigma_0^{1/2} \tau} \\ - \left(6\rho_{i0}^2 - 4\rho_r^2 - 3 \frac{\rho_{i0}''}{\rho_{i0}} \right) \left(\frac{\partial}{\partial t} |\nabla \zeta|^2 \Big|_{(s,0)} \right) \frac{\sigma_0^{1/2}}{\rho_r \mu_j} e^{-\mu_j \rho_r \sigma_0^{1/2} \tau}, \quad (106)$$

and the linear operator acting on (ς, ψ) in (105) is precisely the same as the one acting on $(\partial \tilde{\mu} / \partial s, \partial \vartheta / \partial s)$ in (85). By (93), this operator must have a bounded inverse for a sufficiently large σ_0 . To complete the proof of existence, we thus need to show that $G \in L^2(\mathbb{R}_+)$. This, however can be easily shown since by (46) and (87) we have that

$$|F(\tau)| + |\tilde{R}_\rho| \leq C e^{-\gamma \sigma_0^{-1/2} \tau}.$$

The lemma is proved. \square

Remark 3.2. Repeating the procedure outlined in Lemma 3.2 (cf. also Remark 3.1), one can similarly obtain

$$\left\| \frac{\partial \rho_{i1}}{\partial s} \right\|_{C^2(\mathbb{R}_+)} + \left\| \frac{\partial \varphi_{i1}}{\partial s} \right\|_{C^2(\mathbb{R}_+)} + \left\| \frac{\partial^2 \rho_{i1}}{\partial s^2} \right\|_{C^2(\mathbb{R}_+)} + \left\| \frac{\partial^2 \varphi_{i1}}{\partial s^2} \right\|_{C^2(\mathbb{R}_+)} \leq C, \quad (107)$$

and that

$$\left| \frac{\partial^2 \tilde{\rho}_{i1}}{\partial s^2} \right| + \left| \frac{\partial^2 \varphi'_{i1}}{\partial s^2} \right| + \left| \frac{\partial^2 \tilde{\rho}_{i1}}{\partial s^2} \right| + \left| \frac{\partial^2 \varphi'_{i1}}{\partial s^2} \right| \leq C e^{-\gamma \eta}. \quad (108)$$

We can now set

$$\rho_i = \rho_{i0} + \epsilon \rho_{i1} \quad ; \quad \varphi_i = \varphi_{i0} + \epsilon \varphi_{i1} \quad ; \quad v_i = v_{i0} + \epsilon v_{i1}$$

4 Uniform approximation

We can now construct an approximate solution for (5). To this end, let $\Upsilon \in C^\infty(\mathbb{R}, [0, 1])$ denote a cutoff function satisfying

$$\Upsilon(x) = \begin{cases} 1, & x < 1, \\ 0, & x > 2, \end{cases} \quad \text{where } |\Upsilon'| \leq \frac{3}{2}. \quad (109)$$

Further, let $\delta = \epsilon^\iota$ for some $0 < \iota < 1$ and set

$$\phi_0(t, s) = \frac{1}{\epsilon^2} \varphi_i(\tau, s) \Upsilon(t/\delta) \quad (110a)$$

$$\rho_0(t, s) = \rho_o + [\rho_i(\tau, s) - \rho_a(t, s)] \Upsilon(t/\delta), \quad (110b)$$

and

$$\chi_0(t, s) = \chi_o + v_i(\tau, s) \Upsilon(t/\delta). \quad (110c)$$

Here

$$\rho_a(t, s) = [1 - |\nabla \zeta(s, 0)|^2]^{1/2} + \left(\frac{\partial}{\partial t} [1 - |\nabla \zeta(t, s)|^2]^{1/2} \right) \Big|_{t=0} t,$$

while ρ_o and χ_o are as defined in (31).

Proof of Theorem 1. We first note that the constant C in (10) and (11) may depend on ι and Ω .

We use again Banach fixed point theorem. To this end we set $(\rho_1, \phi_1, \chi_1) = (\rho - \rho_0, \epsilon^2[\phi - \phi_0], \epsilon[\chi - \chi_0])$, and $(\phi_0, \chi_0) = (\epsilon^{-2}\tilde{\phi}_0, \epsilon^{-1}\tilde{\chi}_0)$. We then rewrite (6) in the following form

$$\left\{ \begin{aligned} & -\left(\Delta + \frac{1}{\epsilon^2}(1 - 3\rho_0^2 - |\nabla \tilde{\chi}_0|^2)\right)\rho_1 + \frac{2\rho_0}{\epsilon^2} \nabla \tilde{\chi}_0 \cdot \nabla \chi_1 \\ & \quad = -h_1 - \frac{1}{\epsilon^2} \left[|\nabla \chi_1|^2 \rho_0 + 2\rho_1 \nabla \tilde{\chi}_0 \cdot \nabla \chi_1 \right. \\ & \quad \quad \left. + |\nabla \chi_1|^2 \rho_1 + (3\rho_0 + \rho_1)\rho_1^2 \right] \end{aligned} \right. \quad (111a)$$

$$\left\{ \begin{aligned} & \operatorname{div}(\rho_0^2 \nabla \chi_1) + 2 \operatorname{div}(\rho_1 \rho_0 \nabla \tilde{\chi}_0) - \sigma_0 \epsilon \Delta \phi_1 \\ & \quad = -\epsilon \operatorname{div} H_2 - \operatorname{div}(\rho_1^2 \nabla \tilde{\chi}_0) - \operatorname{div}(\rho_1(2\rho_0 + \rho_1) \nabla \chi_1) \end{aligned} \right. \quad (111b)$$

$$\left\{ \begin{aligned} & \sigma_0 \epsilon^2 \Delta \phi_1 - \rho_0^2 \phi_1 - 2\rho_0 \tilde{\phi}_0 \rho_1 = -\epsilon^2 h_3 + \rho_1(2\rho_0 + \rho_1)\phi_1 + \rho_1^2 \tilde{\phi}_0. \end{aligned} \right. \quad (111c)$$

where

$$h_1 = -\Delta \rho_0 - \frac{1}{\epsilon^2}(1 - |\nabla \tilde{\chi}_0|^2 - \rho_0^2)\rho_0 \quad (111d)$$

$$H_2 = \frac{1}{\epsilon} \rho_0^2 \nabla \tilde{\chi}_0 - \sigma_0 \nabla \tilde{\phi}_0 \quad (111e)$$

$$h_3 = \sigma_0 \Delta \tilde{\phi}_0 - \frac{1}{\epsilon^2} \rho_0^2 \tilde{\phi}_0. \quad (111f)$$

We begin by seeking an estimate for $\|h_1\|_2$. Let

$$\Omega_r = \{x \in \Omega \mid d(x, \partial\Omega) \geq r\}$$

By (46), (103), and (108) we have that for all $x \in \Omega_\delta$

$$h_1 = g_1 + \mathcal{O}(e^{-\epsilon^{-(1-\iota)}}),$$

where g_1 is given by (32). Hence, by (33) we have

$$\|h_1\|_{L^2(\Omega_\delta)} \leq C\epsilon^2. \quad (112)$$

Next, we estimate h_1 in a δ -neighborhood of $\partial\Omega$. In this neighborhood we may write

$$h_1 = -\Delta\rho_i - \frac{\rho_i}{\epsilon^2} \left[1 - \rho_i^2 - \epsilon^2 \left| \nabla v_i + \frac{1}{\epsilon} \nabla \zeta_a \right|^2 \right] + \tilde{h}_1, \quad (113)$$

where

$$\begin{aligned} \tilde{h}_1 = & -\Delta(\rho_o - \rho_a) - \frac{\rho_o - \rho_a}{\epsilon^2} [1 - 3\rho_i^2 - 3\rho_i(\rho_o - \rho_a) - (\rho_o - \rho_a)^2 - \epsilon^2 |\nabla(\chi_o + v_i)|^2] \\ & - 2\rho_i \nabla \left(\chi_o - \frac{\zeta_a}{\epsilon} \right) \cdot \nabla \left(v_i + \frac{1}{\epsilon} \zeta_a \right) - \rho_i \left| \nabla \left(\chi_o - \frac{\zeta_a}{\epsilon} \right) \right|^2, \end{aligned}$$

and

$$\zeta_a = \zeta(s, 0) + t \frac{\partial \zeta}{\partial t}(s, 0) + \frac{t^2}{2} \frac{\partial^2 \zeta}{\partial t^2}(s, 0). \quad (114)$$

Since ρ_a and ζ_a are the respective Taylor expansion of $\rho_{out,o}$ and $\chi_{out,0}$ near the boundary, it easily follows that

$$\|\tilde{h}_1\|_{L^\infty(\Omega \setminus \Omega_\delta)} \leq C\epsilon^{-2(1-\iota)}. \quad (115)$$

To complete the estimate of h_1 it remains necessary to bound $h_1 - \tilde{h}_1$ in (113). To this end we note first that

$$\Delta\rho_i = \left(\frac{1}{\mathbf{g}} \frac{\partial}{\partial s} \right)^2 \rho_i + \frac{1}{\epsilon^2} \rho_i'' - \frac{\kappa}{\epsilon \mathbf{g}} \rho_i' = \frac{1}{\epsilon^2} \rho_{i0}'' + \frac{1}{\epsilon} \left[\frac{\kappa}{\mathbf{g}} \rho_{i0}' + \rho_{i1}'' \right] + \mathcal{O}(1).$$

Furthermore, by (110c)

$$\begin{aligned} \epsilon^2 |\nabla \chi_i|^2 = & |\chi_i'|^2 + \frac{\epsilon^2}{\mathbf{g}^2} \left| \frac{\partial \chi_i}{\partial s} \right|^2 = |\chi_{i0}'|^2 + \left| \frac{\partial \zeta(s, 0)}{\partial s} \right|^2 \\ + 2\epsilon \left[\chi_{i0}' \chi_{i1}' + \left(\frac{\partial^2 \zeta}{\partial t \partial s}(s, 0) \tau + \int_\tau^\infty \left[\frac{\partial \chi_{i0}'(\tau', s)}{\partial s} - \frac{\partial^2 \zeta}{\partial t \partial s}(s, 0) \right] d\tau' \right) \frac{\partial \zeta(s, 0)}{\partial s} + \kappa \tau \left| \frac{\partial \zeta(s, 0)}{\partial s} \right|^2 \right] + \mathcal{O}(\delta^2). \end{aligned}$$

Hence, by (42), (41), (98), (46) and (103) we obtain that

$$\left| -\Delta\rho_i - \frac{\rho_i}{\epsilon^2} [1 - \rho_i^2 - \epsilon^2 |\nabla \chi_i|^2] \right| \leq C\epsilon^{-2(1-\iota)} \quad \forall x \in \Omega \setminus \Omega_\delta.$$

Combining the above with (115) and (113) yields

$$\|h_1\|_2 \leq C\epsilon^{\iota/2-2(1-\iota)}. \quad (116)$$

Next, we obtain an estimate of $\|H_2\|_2$, where H_2 is given by (111e). We have that for all $x \in \Omega_\delta$

$$\operatorname{div} H_2 = g_2 + \mathcal{O}(e^{-\epsilon^{-(1-\iota)}}),$$

where g_2 is given by (34). Hence,

$$\|\operatorname{div} H_2\|_{L^2(\Omega_\delta)} \leq C\epsilon^2. \quad (117)$$

Setting

$$H_2 = \rho_i^2 \nabla \chi_i - \sigma_0 \nabla \varphi_i + \tilde{H}_2 \quad (118)$$

in $\Omega \setminus \Omega_\delta$, where

$$\tilde{H}_2 = \rho_0^2 \nabla \left(\chi_o - \frac{\zeta_a}{\epsilon} \right) + [\rho_0^2 - \rho_i^2] \nabla \chi_0,$$

we deduce the estimate

$$\|\tilde{H}_2\|_{L^\infty(\Omega \setminus \Omega_\delta)} \leq C\epsilon^{1-2(1-\iota)}. \quad (119)$$

It follows that

$$|\rho_i^2 \nabla \chi_i - \sigma_0 \Delta \varphi_i| \leq C\epsilon^{1-2(1-\iota)} \quad \forall x \in \Omega \setminus \Omega_\delta,$$

which together with (119) yields

$$\|H_2\|_{L^\infty(\Omega \setminus \Omega_\delta)} \leq C_\iota \epsilon^{1+\iota/2-2(1-\iota)}. \quad (120)$$

Furthermore, with the aid of (117) we obtain that

$$\|\operatorname{div} H_2\|_2 \leq C_\iota \epsilon^{\iota/2-2(1-\iota)}. \quad (121)$$

Finally, we derive an estimate for $\|h_3\|_2$, where h_3 is given by (111f). By (110) we readily obtain that for $x \in \Omega_\delta$ we have

$$h_3 = \mathcal{O}(e^{-\epsilon^{-(1-\iota)}}).$$

In $\Omega \setminus \Omega_\delta$ it can be easily verified that

$$h_3 = \sigma_0 \Delta \varphi_i - \rho_i^2 \varphi_i + (\rho_o - \rho_a)(\rho_o - \rho_a + 2\rho_i) \varphi_i.$$

Once again, the argument, similar to the one used to derive (116), gives

$$\|h_3\|_2 \leq C_\iota \epsilon^{\iota/2-2(1-\iota)}. \quad (122)$$

Next, let \mathcal{H} be defined by

$$\mathcal{H} = \{(\eta, \omega, \varphi) \in H^2(\Omega, \mathbb{R}^3) \mid (\nabla \eta, \nabla \omega, \nabla \varphi) \cdot \mathbf{n}|_{\partial\Omega} = 0; (\omega)_\Omega = 0\}.$$

We equip \mathcal{H} with the norm

$$\|(\eta, \varphi, \omega)\|_{\mathcal{H}} = \|\eta\|_{\infty} + \|\eta\|_{1,2} + \|\varphi\|_{1,2} + \|\omega\|_{1,2} + \epsilon(\|D^2\eta\|_2 + \|D^2\varphi\|_2 + \|D^2\omega\|_2). \quad (123)$$

Suppose that $(\eta, \omega, \varphi) \in \mathcal{H}$ is a solution of

$$\begin{cases} -\left(\Delta - \frac{1}{\epsilon^2}(3\rho_0^2 - 1 + |\nabla\tilde{\chi}_0|^2)\right)\eta + \frac{2\rho_0}{\epsilon^2}\nabla\tilde{\chi}_0 \cdot \nabla\omega = f_1 & \text{in } \Omega, \end{cases} \quad (124a)$$

$$\begin{cases} -\operatorname{div}(\rho_0^2\omega) - 2\operatorname{div}(\eta\rho_0\nabla\tilde{\chi}_0) + \sigma_0\epsilon\Delta\varphi = f_2 & \text{in } \Omega, \end{cases} \quad (124b)$$

$$\begin{cases} -\sigma_0\epsilon^2\Delta\varphi + \rho_0^2\varphi + 2\rho_0\tilde{\phi}_0\eta = f_3 & \text{in } \Omega, \end{cases} \quad (124c)$$

where $(f_1, f_2, f_3) \in L^2(\Omega, \mathbb{R}^3)$. We split the remainder of the proof into three steps.

Step 1: Prove that $v = (\eta, \omega, \varphi)$ is well-defined. To this end we use the Lax-Milgram lemma. Let $w = (\tilde{\eta}, \tilde{\omega}, \tilde{\varphi})$. Then define the bilinear form $B : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$

$$\begin{aligned} B[v, w] = & \langle \nabla\eta, \nabla\tilde{\eta} \rangle + \frac{1}{\epsilon^2} \left\langle \left(3\rho_0^2 - 1 + |\nabla\tilde{\chi}_0|^2\right)\eta, \tilde{\eta} \right\rangle + \frac{2}{\epsilon^2} \langle \rho_0\nabla\omega, \tilde{\eta}\nabla\tilde{\chi}_0 \rangle + \\ & \frac{1}{\epsilon^2} \left[\langle \rho_0\nabla\omega, \rho_0\nabla\tilde{\omega} \rangle + 2\langle \rho_0\nabla\omega, \eta\nabla\tilde{\chi}_0 \rangle - \sigma_0\epsilon \langle \nabla\tilde{\omega}, \nabla\varphi \rangle + \right. \\ & \left. C_0(\sigma_0\epsilon^2 \langle \nabla\varphi, \nabla\tilde{\varphi} \rangle + \langle \rho_0\tilde{\varphi}, (\rho_0\varphi + 2\eta\tilde{\phi}_0) \rangle) \right], \end{aligned} \quad (125)$$

where $C_0 > 0$ is to be specified later. Since by (110b,c) both $\tilde{\chi}_0$ and $\tilde{\phi}_0$ are in $C^1(\Omega)$, it readily follows from the Sobolev embeddings that there exists $C(\Omega, \epsilon)$ such that

$$|B[v, w]| \leq C\|v\|_{1,2}\|w\|_{1,2}.$$

By (110b) and Lemma 1.1 we have, for a sufficiently small value of ϵ ,

$$\|1 - \rho_0^2\|_{\infty} < \frac{1}{3},$$

and consequently, again for a sufficiently small value of ϵ ,

$$3\rho_0^2 - 1 > 1. \quad (126)$$

We next attempt to estimate $B(v, v)$ from below. Clearly,

$$\begin{aligned} B[v, v] = & \|\nabla\eta\|_2^2 + \frac{1}{\epsilon^2} \|(3\rho_0^2 - 1)^{1/2}\eta\|_2^2 + \frac{1}{\epsilon^2} \left[\|\nabla\tilde{\chi}_0\eta\|_2^2 + \|\rho_0\nabla\omega\|_2^2 \right. \\ & \left. + 4\langle \rho_0\nabla\omega, \nabla\tilde{\chi}_0\eta \rangle - \sigma_0\epsilon \langle \nabla\omega, \nabla\varphi \rangle + C_0(\sigma_0\epsilon^2 \|\nabla\varphi\|_2^2 + \langle \rho_0\varphi, (\rho_0\varphi + 2\eta\tilde{\phi}_0) \rangle) \right]. \end{aligned}$$

First, note that

$$\|\nabla\tilde{\chi}_0\|_{\infty} < \frac{2}{3\sqrt{3}},$$

for ϵ small enough and that from (110a) and Sobolev embeddings

$$\|\eta\tilde{\phi}_0\|_2^2 \leq \|\eta\|_4^2 \|\tilde{\phi}_0\|_4^2 \leq C\epsilon^{1/2} \|\eta\|_{1,2}^2.$$

Furthermore, we have that

$$|\sigma_0 \epsilon \langle \nabla \omega, \nabla \varphi \rangle| \leq \frac{1}{2} (\|\nabla \omega\|_2^2 + \sigma_0^2 \epsilon^2 \|\nabla \varphi\|_2^2),$$

and that

$$\|\nabla \tilde{\chi}_0 \eta\|_2^2 + \|\rho_0 \nabla \omega\|_2^2 + 4 \langle \rho_0 \nabla \omega, \nabla \tilde{\chi}_0 \eta \rangle \geq -3 \|\nabla \tilde{\chi}_0 \eta\|_2^2 \geq -\frac{4}{9} \|\eta\|_2^2.$$

Combining these expressions yields

$$\begin{aligned} B[v, v] &\geq \|\nabla \eta\|_2^2 + \frac{1}{\epsilon^2} \left[\left(1 - \frac{4}{9}\right) \|\eta\|_2^2 + \left(\frac{2}{3} - \frac{1}{2}\right) \|\nabla \omega\|_2^2 \right. \\ &\quad \left. + \sigma_0 \epsilon^2 \left(C_0 - \frac{\sigma_0}{2}\right) \|\nabla \varphi\|_2^2 + \frac{2C_0}{3} \|\varphi\|_2^2 - C \epsilon^{1/4} \|\eta\|_{1,2} \|\varphi\|_2 \right]. \end{aligned}$$

Choosing $C_0 > \sigma_0/2$ we obtain with the aid of Poincaré's inequality that for some $C(\Omega, \sigma_0)$ we have, for sufficiently small ϵ ,

$$|B[v, v]| \geq \frac{C}{\epsilon^2} (\|\eta\|_2^2 + \|\omega\|_{1,2}^2 + \|\varphi\|_2^2) + \|\nabla \eta\|_2^2. \quad (127)$$

for some $C(\Omega, \sigma_0)$ when ϵ is sufficiently small. We can thus conclude the existence of a unique $v \in \mathcal{H}$ such that

$$B[v, w] = \langle F, w \rangle \quad \forall w \in \mathcal{H},$$

where $F = (f_1, \epsilon^{-2} f_2, C_0 \epsilon^{-2} f_3)$.

Let $(\rho_1, \chi_1, \phi_1) \in \mathcal{H}$. We set

$$f_1 = h_1 + \frac{1}{\epsilon^2} \left[|\nabla \chi_1|^2 \rho_0 + \nabla \chi_1 \cdot (2 \nabla \tilde{\chi}_0 + \nabla \chi_1) \rho_1 + (3 \rho_0 + \rho_1) \rho_1^2 \right] \quad (128a)$$

$$f_2 = \epsilon h_2 - \operatorname{div} (\rho_1^2 \nabla \tilde{\chi}_0) - \operatorname{div} (\rho_1 (2 \rho_0 + \rho_1) \nabla \chi_1) \quad (128b)$$

$$f_3 = \epsilon^2 h_3 + \rho_1 (2 \rho_0 + \rho_1) \phi_1 + \rho_1^2 \tilde{\phi}_0. \quad (128c)$$

Substituting the above into (124) we can define the operator $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$ by

$$\mathcal{A}(\rho_1, \chi_1, \phi_1) = (\eta, \omega, \varphi).$$

We look for a fixed point of \mathcal{A} .

Step 2: Let $v = (\rho_1, \chi_1, \phi_1)$. We prove that for sufficiently small ϵ there exists $C(\Omega, \sigma)$ and $r(\epsilon) \leq C \epsilon^{5/4}$ for which

$$v \in B(0, r) \Rightarrow \mathcal{A}(v) \in B(0, r). \quad (129)$$

Let then $0 < r \leq \epsilon$ and $v \in B(0, r)$. We begin by obtaining a bound on $\|\eta\|_2$ and $\|A(v)\|_{1,2}$. By (124a), (110), and (126) we have that

$$\|f_1\|_2 \leq \|h_1\|_2^2 + \frac{1}{\epsilon^2} \left[\|\nabla \chi_1\|_4^2 + \|\rho_1\|_4 (\|\nabla \chi_1\|_4 + (\|\nabla \chi_1\|_8^2) \|\rho_1\|_4^2 + \|\rho_1\|_6^3) \right].$$

By (116) we then obtain

$$\|f_1\|_2 \leq C_s \left(\epsilon^{s/2} + \frac{r^2}{\epsilon^2} \right), \quad (130)$$

where

$$s = \frac{\iota}{2} - 2(1 - \iota).$$

Similarly, we obtain that

$$\|f_2\|_2 \leq C \left[\epsilon \|\operatorname{div} H_2\|_2 + \|\rho_1\|_4^2 + \|\rho_1\|_\infty \|\nabla \rho_1\|_2 + (\|\rho_1\|_\infty^2 + \|\rho_1\|_\infty) \|\Delta \chi_1\|_2 + (1 + \|\rho_1\|_\infty) \|\nabla \rho_1\|_4 \|\nabla \chi_1\|_4 \right].$$

Note, that since $\|v\|_{\mathcal{H}} \leq r$, we have by (123) that $\|\Delta \chi_1\|_2 \leq r/\epsilon$. To bound $\|\nabla \rho_1\|_4$ we use a standard interpolation inequality from (cf. for instance Corollary 4.5 in [19]) from which we conclude that

$$\|\nabla \rho_1\|_4 \leq \|\nabla \rho_1\|_2^{1/2} \|D^2 \rho_1\|_2^{1/2} \leq C \frac{r}{\epsilon^{1/2}}.$$

A similar estimate holds for $\|\nabla \chi_1\|_4$, and hence, with the aid (121) we find that

$$\|f_2\|_2 \leq C_s \left(\epsilon^{1+s/2} + \frac{r^2}{\epsilon} \right). \quad (131)$$

For later reference we need also the following estimate

$$|\langle \omega, f_2 \rangle| \leq C \left[\epsilon \|H_2\|_{L^2(\Omega \setminus \Omega_\delta)} \|\nabla \omega\|_2 + \epsilon \|\operatorname{div} H_2\|_{L^2(\Omega_\delta)} \|\omega\|_2 + [\|\rho_1\|_4^2 + (\|\rho_1\|_\infty + \|\rho_1\|_\infty^2) \|\nabla \chi_1\|_2] \|\nabla \omega\|_2 \right],$$

which leads to

$$|\langle \omega, f_2 \rangle| \leq C_s (\epsilon^{2+s/2} + r^2) \|\omega\|_{1,2}. \quad (132)$$

Finally, by (128c)

$$\|f_3\|_2 \leq C \{ \epsilon^2 \|h_3\|_2 + (\|\rho_1\|_4 + \|\rho_1\|_8^2) \|\phi_1\|_4 + \|\rho_1\|_4^2 \},$$

which leads, by (122), to a similar estimate

$$\|f_3\|_2 \leq C_s \left(\epsilon^{2+s/2} + r^2 \right). \quad (133)$$

Combining the above with (130) and (132) yields

$$|\langle \mathcal{A}(v), F \rangle| \leq C_s \left(\epsilon^{s/2} + \frac{r^2}{\epsilon^2} \right) (\|\eta\|_2 + \|\omega\|_{1,2} + \|\varphi\|_2). \quad (134)$$

As $B(\mathcal{A}(v), \mathcal{A}(v)) = \langle \mathcal{A}(v), F \rangle$ we obtain by (127) that

$$\|\eta\|_2 + \|\omega\|_{1,2} + \|\varphi\|_2 \leq C_s [\epsilon^{2+s/2} + r^2]. \quad (135)$$

and that

$$\|\nabla \eta\|_2 \leq \frac{C_s}{\epsilon} [r^2 + \epsilon^{2+s/2}]. \quad (136)$$

To complete the proof of (129) we first rewrite (124c) in the form

$$\sigma_0 \epsilon^2 \Delta \varphi = \rho_0^2 \varphi + 2\rho_0 \tilde{\phi}_0 \eta - f_3 \text{ in } \Omega,$$

from which we easily conclude, with the aid of (135) and (133), that

$$\|\Delta \varphi\|_2 \leq C_s \left[\frac{r^2}{\epsilon^2} + \epsilon^{s/2} \right].$$

By standard elliptic estimates we have that

$$\|D^2 \varphi\|_2 \leq C_s(\Omega, \sigma_0, j_r) \left[\frac{r^2}{\epsilon^2} + \epsilon^{s/2} \right]. \quad (137)$$

In a similar manner we rewrite (124b) in the form

$$-\operatorname{div}(\rho_0^2 \omega) = 2 \operatorname{div}(\eta \rho_0 \nabla \tilde{\chi}_0) - \sigma_0 \epsilon \Delta \varphi + f_2$$

As in the proof of (137) we then obtain that

$$\|\operatorname{div}(\rho_0^2 \nabla \omega)\|_2 \leq C_s \left[\frac{r^2}{\epsilon} + \epsilon^{1+s/2} \right].$$

Thus, by (135) and (110b)

$$\|\Delta \omega\|_2 \leq \|\operatorname{div}(\rho_0^2 \nabla \omega)\|_2 + \|\nabla \rho_0\|_\infty \|\nabla \omega\|_2 \leq C_s \left[\frac{r^2}{\epsilon} + \epsilon^{1+s/2} \right],$$

from which, by standard elliptic estimates we obtain that

$$\|D^2 \omega\|_2 \leq C_s \left[\frac{r^2}{\epsilon} + \epsilon^{1+s/2} \right]. \quad (138)$$

To complete the proof we need yet to bound $\epsilon \|D^2 \eta\|_2$ and $\|\eta\|_\infty$. To this end we rewrite (124a) in the form

$$-\Delta \eta = -\frac{1}{\epsilon^2} (|\nabla \tilde{\chi}_0|^2 - 3\rho_0^2 + 1) \eta - \frac{2}{\epsilon^2} \rho_0 \nabla \tilde{\chi}_0 \cdot \nabla \omega + f_1.$$

It easily follows that

$$-\frac{1}{\epsilon^2} \|(|\nabla \tilde{\chi}_0|^2 - 3\rho_0^2 + 1) \eta\|_2 \leq \frac{C}{\epsilon^2} (r^2 + \epsilon^2). \quad (139)$$

Furthermore, as

$$\|\rho_0 \nabla \tilde{\chi}_0 \cdot \nabla \omega\|_2 \leq C r^2,$$

we obtain with the aid of (139) and (130) that

$$\|D^2 \eta\|_2 \leq C_s \left[\frac{r^2}{\epsilon^2} + \epsilon^{s/2} \right], \quad (140)$$

We can now employ Agmon's inequality (cf. [20, Lemma 13.2]) from which we learn, using the above in conjunction with (135), that

$$\|\eta\|_\infty \leq \frac{C_s}{\epsilon} [r^2 + \epsilon^{2+s/2}]. \quad (141)$$

Combining (141) with (135), (136), (140), (137), and (138) yields

$$\|\mathcal{A}(v)\|_{\mathcal{H}} \leq C_s \left[\frac{r^2}{\epsilon} + \epsilon^{1+s/2} \right].$$

We may thus choose $r = \epsilon^{5/4}$ to obtain that

$$\|\mathcal{A}(v)\|_{\mathcal{H}} \leq C\epsilon^{1+s/2} < r$$

for sufficiently small ϵ and δ and $1/2 < s < 1$.

Step 3: Let $(v_1, v_2) \in B(0, r)^2$. We prove that there exists a $\gamma < 1$ such that

$$\|\mathcal{A}(v_1) - \mathcal{A}(v_2)\|_{\mathcal{H}} \leq \gamma \|v_1 - v_2\|_{\mathcal{H}}. \quad (142)$$

It can be easily verified that

$$\begin{aligned} \|f_1(v_1) - f_1(v_2)\|_2 &\leq C \left(\|J\|^2 + \frac{1}{\epsilon^2} \right) r \|v_1 - v_2\|_{\mathcal{H}}, \\ \|f_2(v_1) - f_2(v_2)\|_2 &\leq Cr \|v_1 - v_2\|_{\mathcal{H}}, \\ \|f_2(v_1) - f_2(v_2)\|_2 &\leq Cr \|v_1 - v_2\|_{\mathcal{H}}. \end{aligned}$$

Let now $\mathcal{A}(v_1) = (\eta_1, \omega_1, \varphi_1)$ and $\mathcal{A}(v_2) = (\eta_2, \omega_2, \varphi_2)$. As

$$B(\mathcal{A}(v_1) - \mathcal{A}(v_2), \mathcal{A}(v_1) - \mathcal{A}(v_2)) = \langle \mathcal{A}(v_1) - \mathcal{A}(v_2), F(v_1) - F(v_2) \rangle,$$

we obtain by (127) that

$$\|\eta_1 - \eta_2\|_2 \leq Cr \|v_1 - v_2\|_{\mathcal{H}}.$$

In the same manner used to derive (135) and (136) we can now obtain that

$$\|\omega_1 - \omega_2\|_{1,2} + \|\varphi_1 - \varphi_2\|_{1,2} \leq Cr \|v_1 - v_2\|_{\mathcal{H}},$$

and that

$$\|\nabla(\eta_1 - \eta_2)\|_2 \leq C \frac{r}{\epsilon} \|v_1 - v_2\|_{\mathcal{H}}.$$

We then proceed in precisely the same manner as in the derivation of (137) and (138) to obtain that

$$\epsilon \|\omega_1 - \omega_2\|_{2,2} + \|\varphi_1 - \varphi_2\|_{2,2} \leq Cr \|v_1 - v_2\|_{\mathcal{H}}.$$

Finally, using the same procedure as in the derivation of (140) and (141) gives

$$\|\eta_1 - \eta_2\|_\infty + \epsilon^2 \|\eta_1 - \eta_2\|_{2,2} \leq C \frac{r}{\epsilon} \|v_1 - v_2\|_{\mathcal{H}}.$$

Combining all of the above then yields

$$\|\mathcal{A}(v_1) - \mathcal{A}(v_2)\|_{\mathcal{H}} \leq C \frac{r}{\epsilon} \|v_1 - v_2\|_{\mathcal{H}},$$

and since $r = \delta\epsilon$, we obtain (142) for a sufficiently small value of δ . \square

5 Linear stability

In this section we examine the linear stability of the solution we have obtained in the previous section. To this end we define the non-linear operator $\mathcal{L} : \mathcal{W} \rightarrow L^2(\Omega, \mathbb{C})$ where

$$\mathcal{U} = \{u \in H^2(\Omega, \mathbb{C}) \mid \partial u / \partial \mathbf{n}|_{\partial\Omega} = 0\},$$

be given, for any $u \in U_g$

$$\mathcal{L}u = -\Delta u + i\phi u - \frac{u}{\epsilon^2}(1 - |u|^2)$$

In the above ϕ denotes a non-local, non-linear operator acting on u . In view of (3) we define this operator as the solution of

$$\begin{cases} -\sigma \Delta \phi = \operatorname{div}(\Im\{\bar{u} \nabla u\}) & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \mathbf{n}} = -\frac{J}{\sigma} & \text{on } \partial\Omega, \\ \int_{\Omega} |u|^2 \phi \, dx = 0. \end{cases}$$

The system (2) can then be written in the form

$$u_t + \mathcal{L}u = 0. \quad (143)$$

Note that the integral condition on ϕ must be satisfied by any root $u_s = \rho_s e^{is} \in \mathcal{U}$ of \mathcal{L} , or equivalently by any steady state solution of (143). It can be easily verified that one can derive it from the fact that $\Im\langle u_s, \mathcal{L}u_s \rangle = 0$.

We look for the spectrum of $\mathcal{A} = D\mathcal{L}(u_s)$, or the Frechet derivative of \mathcal{L} at u_s . Set $\phi_s = \phi(u_s)$. It can be readily verified that

$$\mathcal{A}u = -\Delta u + i(\phi_s u + \hat{\varphi} u_s) - \frac{u}{\epsilon^2}(1 - \rho_s^2) + \frac{2u_s}{\epsilon^2} \Re(\bar{u}_s u), \quad (144)$$

where $\hat{\varphi}(u, u_s)$ is a non-local linear operator given by the solution of

$$\begin{cases} -\sigma \Delta \hat{\varphi} = \operatorname{div}(\Im\{\bar{u}_s \nabla u + \bar{u} \nabla u_s\}) & \text{in } \Omega \\ \frac{\partial \hat{\varphi}}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \\ \int_{\Omega} [|u_s|^2 \hat{\varphi} + 2\phi_s \Re(\bar{u}_s u)] \, dx = 0. \end{cases} \quad (145)$$

Note that \mathcal{A} has a non-trivial kernel, that is $\mathcal{A} i u_s = 0$. This non-trivial kernel reflects the fact that

$$e^{-i\Theta} \mathcal{L} e^{i\Theta} = \mathcal{L}.$$

Let $\{u_n\}_{n=0}^{\infty}$ denote the system of eigenfunctions associated with \mathcal{A} where $u_0 = i u_s$. Since by Theorem 16.5 in [20] we have $\operatorname{span}\{u_n\}_{n=0}^{\infty} = L^2(\Omega, \mathbb{C})$ we can set

$$D(\mathcal{A}) = \mathcal{U} \cap \overline{\operatorname{span}\{u_n\}_{n=1}^{\infty}}$$

as the domain of \mathcal{A} , thereby eliminating $i u_s$ from the domain.

Let $\tilde{u} = \tilde{\rho}e^{i\tilde{\chi}} \in \mathcal{U}$ denote an infinitesimal perturbation of u_s . One can use the representation $\tilde{u} = u_s + \delta' u$ where δ' is a small parameter, or equivalently the representation $\tilde{u} = (\rho_s + \delta' \rho)e^{i(\chi_s + \delta' \chi)}$. It can be easily verified that as $\delta \rightarrow 0$ we obtain

$$u = e^{i\chi_s}(\rho + i\rho_s\chi).$$

Let then

$$\mathcal{B} = e^{-i\chi_s} \mathcal{A} e^{i\chi_s},$$

which is defined on $D(\mathcal{B}) = e^{-i\chi_s} D(\mathcal{A})$. Since $e^{i\chi_s}$ is a unitary operator we have $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$. We next write

$$\mathcal{B} = -\Delta + |\nabla \chi_s|^2 - \frac{1}{\epsilon^2}(1 - \rho_s^2) + \frac{2}{\epsilon^2}\rho_s^2 \Re + i(-\Delta \chi_s - 2\nabla \chi_s \cdot \nabla + \phi_s + \rho_s \varphi). \quad (146)$$

In the above $\varphi = \hat{\varphi}e^{i\chi_s}$. For any $v \in D(\mathcal{A})$ we have that $\varphi(v)$ is given by the solution

$$\begin{cases} -\sigma \Delta \varphi = \operatorname{div}(\rho_s \nabla \Im v + 2\rho_s \nabla \chi_s \Re v) & \text{in } \Omega \\ \frac{\partial \varphi}{\partial \mathbf{n}} = 0 & \text{on } \partial \Omega \\ \int_{\Omega} [\rho_s^2 \varphi + 2\phi_s \rho_s \Re v] dx = 0. \end{cases} \quad (147)$$

In the sequel we look for non-trivial solution for the system

$$\Re(\mathcal{B}v) = \Re(\lambda v) \quad (148a)$$

$$\Im(\mathcal{B}v) = \Im(\lambda v). \quad (148b)$$

We consider here the stability, in a one-dimensional setting, of the following solution of (6a)-(6c)

$$\rho_s = 1 - \beta^2 \quad ; \quad \chi_s = \frac{\beta}{\epsilon} \left(x - \frac{l}{2} \right) \quad ; \quad \phi_s = 0.$$

In this case $\Omega = (0, l)$, and the Frechet derivative of \mathcal{L} near $u_s = e^{i\chi_s}$ is given in the form

$$D\mathcal{L}u = -u'' - \frac{\beta^2}{\epsilon^2}u + 2\frac{\sqrt{1-\beta^2}}{\epsilon^2}u_s \Re(e^{-i\chi_s}u) + iu_s \varphi$$

where

$$-\sigma \varphi' + 2\frac{\beta \sqrt{1-\beta^2}}{\epsilon} \Re(e^{-i\chi_s}u) + \sqrt{1-\beta^2} [\Im(e^{-i\chi_s}u)]' = 0.$$

Let $u = ve^{i\chi_s}$. In the absence of a boundary layer, we choose the domain of $\mathcal{B} = e^{i\chi_s} D\mathcal{L}e^{-i\chi_s}$ to be given by

$$D_l(\mathcal{B}) = \{v \in H^2([0, l], \mathbb{C}) \mid v_r(0) = v_r(l) = v'_i(0) = v'_i(l) = 0; \int_0^l v_i dx = 0\},$$

where $v_r = \Re v$ and $v_i = \Im v$.

We can now state

Proposition 4. *Let $\mathcal{T} = e^{-t\mathcal{B}}$ denote the semigroup associated with $-\mathcal{B}$. Then, \mathcal{T} is stable if and only if $0 \leq \beta \leq 1/\sqrt{3}$.*

Proof. We search for eigenvalues of $\mathcal{B} : D(\mathcal{B}) \rightarrow L^2(\mathbb{R}^2)$. The eigenvalue problem we obtain is

$$\begin{cases} -\left(\frac{\partial}{\partial x} + i\frac{\beta}{\epsilon}\right)^2 v - \frac{\beta^2}{\epsilon^2} v + 2\frac{1-\beta^2}{\epsilon^2} v_r + i\sqrt{1-\beta^2}\varphi = \lambda v & 0 \leq x \leq l \\ -\sigma\varphi' + 2\frac{\beta\sqrt{1-\beta^2}}{\epsilon} v_r + \sqrt{1-\beta^2} v'_i = 0 & 0 \leq x \leq l \\ v \in D(\mathcal{B}). \end{cases}$$

Applying the transformation $(x, \varphi, \sigma) \rightarrow (x/\epsilon, \epsilon^2\varphi, \sigma/\epsilon^2)$ we obtain the system

$$\begin{cases} \mathcal{P}v \stackrel{\text{def}}{=} -\left(\frac{\partial}{\partial x} + i\beta\right)^2 v - \beta^2 v + 2(1-\beta^2)v_r + i\sqrt{1-\beta^2}\varphi = \lambda v & 0 < x < \frac{l}{\epsilon} \\ -\sigma\varphi' + 2(\beta\sqrt{1-\beta^2})v_r + \sqrt{1-\beta^2}v'_i = 0 & 0 < x < \frac{l}{\epsilon} \\ v \in D_{l/\epsilon}(\mathcal{B}). \end{cases} \quad (149)$$

We next substitute the ansatz

$$v = \sin(\gamma x) + iA \cos(\gamma x), \quad (150)$$

where $\gamma = \gamma_n = \epsilon n\pi/l$, for any $n \in \mathbb{N}$. By the second equation of (149) we then have

$$\varphi = \frac{\sqrt{1-\beta^2}}{\sigma} \left[-2\frac{\beta}{\gamma} + A \right] \cos(\gamma x).$$

Assuming $\lambda \in \mathbb{R}$ we then obtain

$$[\gamma^2 - 2A\gamma\beta + 2(1-\beta^2) - \lambda] \sin(\gamma x) + \left\{ A\gamma^2 - 2\gamma\beta + \frac{1-\beta^2}{\sigma} \left[-2\frac{\beta}{\gamma} + A \right] - A\lambda \right\} \cos(\gamma x) = 0. \quad (151)$$

To have a nontrivial solution we must then have

$$A[\gamma^2 - 2A\gamma\beta + 2(1-\beta^2)] = A\gamma^2 - 2\gamma\beta + \frac{1-\beta^2}{\sigma} \left[-2\frac{\beta}{\gamma} + A \right],$$

yielding two possible values for A ,

$$A_{\pm} = \frac{1-\beta^2}{2\beta\gamma} \left(1 - \frac{1}{2\sigma} \right) \pm \frac{\sqrt{(1-\beta^2)^2 \left(1 - \frac{1}{2\sigma} \right)^2 + 4\beta^2 \left(\gamma^2 + \frac{1-\beta^2}{\sigma} \right)}}{2\beta\gamma} \quad (152)$$

By (151) and (152) we then have

$$\lambda_{\pm} = \gamma^2 + (1-\beta^2) \left(1 + \frac{1}{2\sigma} \right) \pm \sqrt{(1-\beta^2)^2 \left(1 - \frac{1}{2\sigma} \right)^2 + 4\beta^2 \left(\gamma^2 + \frac{1-\beta^2}{\sigma} \right)}. \quad (153)$$

Note that λ_{\pm} remain real and distinct for all σ , which is in line with the PT symmetry of \mathcal{B} and the fact that it becomes self-adjoint as $\sigma \rightarrow \infty$. As $\gamma \rightarrow 0$ (or as $\epsilon \rightarrow 0$), we obtain

$$\lambda_{\pm}^0 \lim_{\gamma \rightarrow 0} \lambda_{\pm} = (1-\beta^2) \left(1 + \frac{1}{2\sigma} \right) \pm \sqrt{(1-\beta^2)^2 \left(1 + \frac{1}{2\sigma} \right)^2 - \frac{2(1-\beta^2)}{\sigma} (1-3\beta^2)}.$$

It readily follows from here that there exists $\lambda_-^0 < 0$ if and only if (assuming (150)) $\beta \leq 1/\sqrt{3}$ (a result which is stated without proof in [4]).

We note that the system of eigenfunctions

$$v_0 = i \quad ; \quad v_n = \{\sin(\gamma_n x) + iA_{\pm}^n \cos(\gamma_n x)\}_{n=1}^{\infty} ,$$

is complete in $L^2([0, l/\epsilon], \mathbb{C})$ for every $0 \leq \beta < 1$, since by (152) $A_+^n \neq A_-^n$ for all $n \in \mathbb{N}$. Hence, we can form from the above system, by simple linear combinations, the system

$$\{\sin(\gamma_n x)\}_{n=1}^{\infty} \quad ; \quad \{i \cos(\gamma_n x)\}_{n=0}^{\infty} ,$$

which is clearly complete, inasmuch as Fourier series are complete. It follows, therefore, that $e^{-t\mathcal{B}}$ is stable for all $\sigma \in \mathbb{R}$ and $\beta \leq 1/\sqrt{3}$. \square

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